An Approximation Theorem for the Spectrum of Schrödinger Operators Related to Quasicrystals

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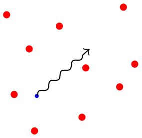
Motivation

physical:



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mathematical:

spectrum of the
Schrödinger operator

periodic: Floquet-Bloch theory

aperiodic: ???

An approximation theorem

Theorem

Let $\Xi \subseteq \mathcal{A}^{\mathbb{Z}}$ be a periodically approximable subshift. Consider a family of selfadjoint, linear, bounded Schrödinger operators

$$H_\xi : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \qquad \xi \in \Xi.$$

For $\xi \in \Xi$ there exists a sequence of periodic Schrödinger operators $H_n : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $n \in \mathbb{N}$ such that:

(i) *The sequence $(H_n)_{n\in\mathbb{N}}$ converge strongly to H_ξ when $n \to \infty$.*

(ii) *The sequence of spectra converge, i.e. $\sigma(H_n) \stackrel{n\to\infty}{\longrightarrow} \bigcup_{\xi\in\Xi} \sigma(H_\xi)$ with respect to the Hausdorff metric.*

- What is a Schrödinger operator $H := (H_\xi)_{\xi\in\Xi}$ related to Ξ?
- What are the conditions for convergence of the spectrum?
- Is there a constructive way to get the periodic approximants?

Subshifts and dictionaries

- finite set $\mathcal{A}$ (alphabet)
- $\mathcal{A}^{\mathbb{Z}} := \{\xi : \mathbb{Z} \to \mathcal{A}\}$ product topology (compact space)
- $\mathcal{A}^{m} := \{u : \{1, \dots, m\} \to \mathcal{A}\}$
- dictionary $\mathcal{W}(\xi) := \{\text{finite subword of } \xi\}$
- shift $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}, \quad S\xi(j) := \xi(j-1)$
- subshift: $\Xi \subseteq \mathcal{A}^{\mathbb{Z}}$ closed, invariant

Schrödinger operator related to Ξ

- operator = kinetic part + potential part

$$H_\xi \psi(j) := \psi(j-1) + \psi(j+1) \quad + \quad V(S^j \xi) \cdot \psi(j)$$

where $V : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$ continuous, takes finitely many values.

- $(H_\xi)_{\xi \in \Xi}$ covariant, strongly continuous

- more general: groupoid $\Gamma := \Xi \rtimes_S \mathbb{Z}$

with C^*-algebra $\mathcal{C}^*(\Gamma) := \overline{\mathcal{C}_c(\Gamma)}$

- left-regular representation $H_\xi := \pi_\xi(h), \quad \xi \in \Xi \quad$ (convolution)

Conditions on the approximants

- idea: Find $\Xi_n \subseteq \mathcal{A}^{\mathbb{Z}},\ n \in \mathbb{N}$ converging to $\Xi \subseteq \mathcal{A}^{\mathbb{Z}}$
- associated dictionary $\mathcal{W}(\Xi) := \bigcup_{\xi \in \Xi} \mathcal{W}(\xi)$
- $\Xi_n \stackrel{n\to\infty}{\longrightarrow} \Xi \iff$ all $m \in \mathbb{N}$ exists $n_m \in \mathbb{N}$ such that $\mathcal{W}(\Xi_n) \cap \mathcal{A}^m = \mathcal{W}(\Xi) \cap \mathcal{A}^m$ for all $n \geq n_m$
- convergence of the local patterns
- choose Schrödinger operators H_n with respect to Ξ_n in an appropriate relation to H, i.e.

$$H_\eta \psi(j) := \psi(j-1) + \psi(j+1) \ + \ V(S^j \eta) \cdot \psi(j), \qquad \eta \in \Xi_n$$

with the same $V : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$

How to get periodic approximants?

- Ξ subshift that is periodically approximable
- associate graphs: Gähler-Anderson-Putnam graphs $\mathcal{G}_n \in \mathbb{N}$,
  de Bruijn graphs (1-dimensions)
- choose closed path in $\mathcal{G}_n$ visiting all vertices

  $\rightsquigarrow$ $\xi_n \in \mathcal{A}^{\mathbb{Z}}$ periodic such that $\mathcal{W}(\xi_n) \cap \mathcal{A}^n = \mathcal{W}(\Xi) \cap \mathcal{A}^n$

  $\rightsquigarrow$ $\Xi_n := \{S^j \xi_n \ : \ j \in \mathbb{Z}\}$ subshift such that $\mathcal{W}(\Xi_n) \cap \mathcal{A}^n = \mathcal{W}(\Xi) \cap \mathcal{A}^n$

Thank you for your attention!