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# **The connection of the K-theory with the Gap-labeling theorem of Schrödinger operators**

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(24th September 2017)



## **Abstract**

This notes is based on a lecture given at the Israel Institute of Technology in Haifa in August and September 2017. The reader is invited to send comments and remarks to the author to improve this notes.



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# 1 Introduction

## 1.1 Motivation - Schrödinger operators

- solid state physics: long time behavior of particle
- ⇒ (QM) study spectral theory of Schrödinger operators

$$H := -\Delta + V \quad (\text{self-adjoint})$$

continuous model

$H$  unbounded

Example on  $L^2(\mathbb{R})$

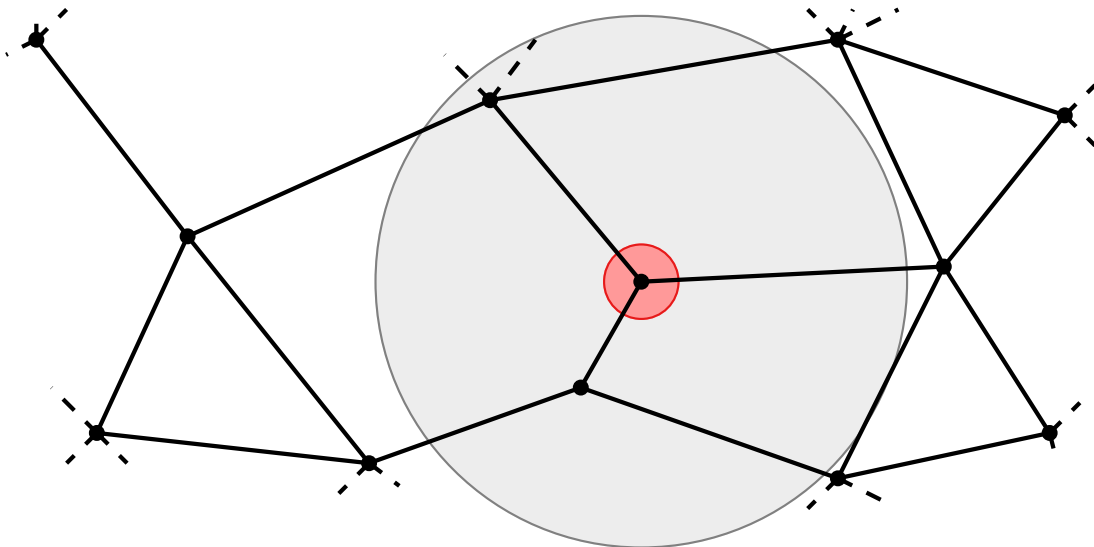
$$H = -\frac{d^2}{dx^2} + V$$

discrete model

$H$  bounded (depends on vertex degree)

Example on  $\ell^2(\mathbb{Z})$

$$\begin{aligned} \frac{d}{dx}\psi(x_0) &\approx \frac{\psi(x_0+h) - \psi(x_0)}{h} \\ \frac{d^2}{dx^2}\psi(x_0) &\approx \frac{(\psi(x_0+h) - \psi(x_0)) - (\psi(x_0+2h) - \psi(x_0+h))}{h^2} \\ &\stackrel{h=1}{=} -\psi(x_0) - \psi(x_0+2) + 2 \cdot \psi(x_0+1) \\ (H\psi)(n) &:= \psi(n-1) + \psi(n+1) + V(n)\psi(n) \end{aligned}$$



## 1.2 The integrated density of states (IDS)

- the following approach has been widely analyzed as discussed later, see e.g. [Bel92, Len02, LS05, LMV08, Ele08, LV09, LSV11, Pog14, PS16a]

- consider the Schrödinger operator  $H : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  defined by

$$(H\psi)(n) := \psi(n-1) + \psi(n+1) + V(n)\psi(n)$$

- take an exhausting sequence  $F_N := \{-N, -N+1, \dots, N\} \subseteq \mathbb{Z}, N \in \mathbb{N}$ , and denote by  $H_{F_N}$  the restriction of  $H$  to  $F_N$  with Dirichlet boundary conditions
- consider

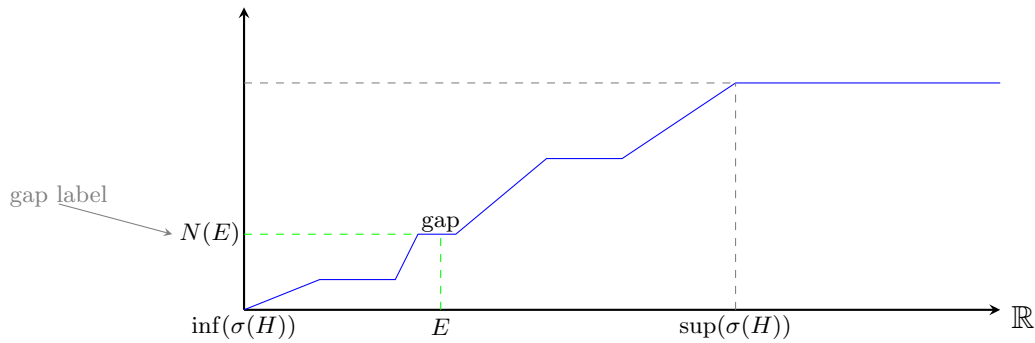
$$N_{F_N}(E) := \#\{\lambda \in \mathbb{R} \mid \lambda \text{ eigenvalue of } H_{F_N} \text{ and } \lambda \leq E\}$$

**Definition 1.1.** *The limit (if it exists)*

$$N(E) := \lim_{N \rightarrow \infty} \frac{N_{F_N}(E)}{\#F_N}$$

is called integrated density of states (IDS) of  $H$ .

- $\chi(H_{F_N} \leq E)$  is the eigenprojection onto the eigenspace of  $H_{F_N}$  with energies less or equal than  $E$
- then  $N_{F_N}(E) = \text{tr}(\chi(H_{F_N} \leq E))$



- People discovered in examples that the spectral gaps can be labeled such that the labeling is stable under small perturbations of the Hamiltonian.
- Based on this experiences Jean Bellissard realized that the Gap labeling should be of topological nature. Thus, he connected the gap labels with the  $K$ -theory ( $K_0$ -group) and the trace of associated  $C^*$ -algebras.

**Aim:** Determine the gap labels of a given Schrödinger operator.

### 1.3 Strategy

- dynamical approach  $\rightarrow$  view operators as suitable integral operators with kernels on the dynamical system ( $C^*$ -algebra approach)
- Pastur-Shubin formula: write  $N(E)$  as a trace of the corresponding eigenprojections
- under suitable ergodicity assumptions the trace is given by an integral over the (unique) ergodic measure
- define a group structure on the "equivalence classes" (by unitary) of the eigenprojection ( $K_0$ -group)
- then the possible gap labels are contained in the image of the trace of the  $K_0$  group

## 2 $C^*$ -algebras associated with dynamical systems and the integrated density of states

### 2.1 $C^*$ -algebras

In the following section, fundamental notions of  $C^*$ -algebras are introduced. This is just a short summary. The reader is referred to [Dix77, Dix81, Mur90, Bla17] and references therein for further background.

**Definition 2.1** (algebra). *An algebra  $\mathfrak{A}$  is a vector space (over  $\mathbb{C}$ ) with multiplication  $\star : \mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$ ,  $(a, b) \mapsto a \star b$ , satisfying*

- $a \star (b \star c) = (a \star b) \star c$  *(associative)*
- $(a + b) \star c = a \star c + b \star c$  *(distributive)*
- $a \star (b + c) = a \star b + a \star c$
- $\alpha \cdot (a \star b) = (\alpha \cdot a) \star b = a \star (\alpha \cdot b)$

for all  $a, b, c \in \mathfrak{A}$  and  $\alpha \in \mathbb{C}$ . An algebra  $A$  is called unital if there is an  $e \in \mathfrak{A}$  such that  $e \star a = a \star e = a$  for all  $a \in \mathfrak{A}$ . Then  $e$  is called unit.

**Remark 2.2.** *If  $\mathfrak{A}$  is a unital algebra, then the unit  $e$  is unique. ( $e = e \star e' = e'$ ) It is worth mentioning that the multiplication is not commutative, in general. An algebra  $\mathfrak{A}$  is said to be commutative if  $a \star b = b \star a$  for all  $a, b \in \mathfrak{A}$  and otherwise  $A$  is noncommutative.*

**Definition 2.3** (Banachalgebra). *A tuple  $(\mathfrak{A}, \|\cdot\|)$  is called a normed algebra if  $\mathfrak{A}$  is an algebra and the map  $\|\cdot\| : \mathfrak{A} \rightarrow [0, \infty)$  is a norm satisfying  $\|a \star b\| \leq \|a\| \|b\|$  for all  $a, b \in \mathfrak{A}$ . If  $\mathfrak{A}$  is additional unital, we require  $\|e\| = 1$ . Furthermore, a normed algebra  $(\mathfrak{A}, \|\cdot\|)$  is called Banachalgebra if  $(\mathfrak{A}, \|\cdot\|)$  is a complete space.*

*Example 2.4.* The normed space  $(\ell^1(\mathbb{Z}), \|\cdot\|_1)$  with  $\|\psi\|_1 := \sum_{n \in \mathbb{Z}} |\psi(n)|$  and multiplication

$$(\psi \star \varphi)(n) := \sum_{k \in \mathbb{Z}} \psi(n - k) \varphi(k)$$

is a (commutative) Banachalgebra with unit  $\delta_0 \in \ell^1(\mathbb{Z})$  defined by  $\delta_0(n) = 1$  if  $n = 0$  and otherwise  $\delta_0(n) = 0$ .

**Remark 2.5.** *The constraint  $\|a \star b\| \leq \|a\| \|b\|$  guarantees the continuity of the multiplication on  $A$ .*

**Definition 2.6** ( $\ast$ -algebra). *Let  $\mathfrak{A}$  be an algebra. A map  $\ast : \mathfrak{A} \rightarrow \mathfrak{A}$  is called involution if*

- $(a + \alpha b)^\ast = a^\ast + \bar{\alpha} b^\ast$
- $(a \star b)^\ast = b^\ast \star a^\ast$
- $(a^\ast)^\ast = a$



holds for all  $a, b \in \mathfrak{A}$  and  $\alpha \in \mathbb{C}$ . Then  $(\mathfrak{A}, *)$  is called  $*$ -algebra / involutive algebra.

**Definition 2.7** ( $C^*$ -algebra). Let  $(\mathfrak{C}, *, \|\cdot\|)$  be a  $*$ -Banachalgebra. Then  $\mathfrak{C}$  is called a  $C^*$ -algebra if

$$\|a\|^2 \leq \|a^* \star a\|, \quad a \in \mathfrak{C},$$

holds.

**Remark 2.8.** The constraint  $\|a\|^2 \leq \|a^* \star a\|$  is equivalent to  $\|a\|^2 = \|a^* \star a\|$ . For a  $C^*$ -algebra,  $*$  :  $\mathfrak{C} \rightarrow \mathfrak{C}$  is isometric (i.e.,  $\|a^*\| = \|a\|$ ) since

$$\|a\|^2 \leq \|a^* \star a\| \leq \|a^*\| \|a\| \Rightarrow \|a\| \leq \|a^*\|.$$

*Example 2.9* (Complex plane). The set  $\mathfrak{C} = \mathbb{C}$  with pointwise multiplication and involution defined by complex conjugation is a unital (commutative)  $C^*$ -algebra with unit  $e = 1$  where  $\|\alpha\| := |\alpha|$ .

*Example 2.10.* Let  $X$  be a topological space (locally compact). The set  $\mathfrak{C} = \mathcal{C}_0(X)$  with pointwise multiplication, uniform norm  $\|f\|_\infty := \sup_{x \in X} |f(x)|$  and involution defined by complex conjugation is a (commutative)  $C^*$ -algebra. It is unital if and only if  $X$  is a compact space.

*Example 2.11* (Linear bounded operators). Let  $\mathcal{H}$  be a Hilbert space. Let  $\mathcal{L}(\mathcal{H})$  be the set of all linear, bounded operators  $T : \mathcal{H} \rightarrow \mathcal{H}$  with multiplication defined by composition, involution defined by the adjoint of an operator and operator norm  $\|T\| := \sup_{\|\psi\| \leq 1} \|T\psi\|$ . Then  $\mathfrak{C} := \mathcal{L}(\mathcal{H})$  is a (noncommutative)  $C^*$ -algebra. For a normal operator  $T \in \mathcal{L}(\mathcal{H})$ , the norm closure of the set  $\{p(T) \mid p \text{ polynomial}\}$  is a (commutative)  $C^*$ -subalgebra.

*Example 2.12.* The normed space  $(\ell^1(\mathbb{Z}), \|\cdot\|_1)$  with  $\|\psi\|_1 := \sum_{n \in \mathbb{Z}} |\psi(n)|$  and convolution

$$(\psi \star \varphi)(n) := \sum_{k \in \mathbb{Z}} \psi(n-k)\varphi(k)$$

is a (commutative) Banachalgebra with unit  $\delta_0 \in \ell^1(\mathbb{Z})$  defined by  $\delta_0(n) = 1$  if  $n = 1$  and otherwise  $\delta_0(n) = 0$ . The map  $*$  :  $\ell^1(\mathbb{Z}) \rightarrow \ell^1(\mathbb{Z})$  defined by  $\psi^*(n) := \overline{\psi(-n)}$  defines an involution. Hence,  $(\ell^1(\mathbb{Z}), *, \|\cdot\|_1)$  is a  $*$ -Banachalgebra but not a  $C^*$ -algebra: Let  $\psi \in \ell^1(\mathbb{Z})$  be defined by  $\psi(0) = 1, \psi(1) = \psi(2) = -1$  and  $\psi(n) = 0$  if  $n \in \mathbb{Z} \setminus \{0, 1, 2\}$ . Then

$$(\psi^* \star \psi)(n) = \begin{cases} 3, & n = 0, \\ -1, & n = \pm 2, \\ 0, & \text{otherwise} \end{cases}$$

holds. Thus,  $\|\psi^* \star \psi\| = 5$  while  $\|\psi\|^2 = 9$ . Consequently, the  $C^*$ -identity is not satisfied.

**Definition 2.13** (Representation). Let  $\mathfrak{A}$  be a  $*$ -algebra. Then a pair  $(\pi, \mathcal{H})$  is called a  $*$ -representation of  $\mathfrak{A}$  if  $\mathcal{H}$  is a Hilbert space and  $\pi : \mathfrak{A} \rightarrow \mathcal{L}(\mathcal{H})$  is a  $*$ -homomorphism, i.e.,

- (i)  $\pi$  is linear;
- (ii)  $\pi$  is multiplicative, i.e.,  $\pi(f \star g) = \pi(f)\pi(g)$  holds for all  $f, g \in \mathfrak{A}$ ;
- (iii)  $\pi$  preserves the involution, i.e.,  $\pi(f^*) = \pi(f)^*$  holds for all  $f \in \mathfrak{A}$  where  $\pi(f)^*$  is the adjoint operator of  $\pi(f) \in \mathcal{L}(\mathcal{H})$ .

A family of representations  $(\pi^x, \mathcal{H}_x)_{x \in X}$  is called faithful whenever the family is injective, i.e.,  $f = 0$  if and only if  $\pi^x(f) = 0$  for all  $x \in X$ .

**Definition 2.14** (invertible). *Let  $\mathfrak{A}$  be a unital Banachalgebra. An element  $a \in \mathfrak{A}$  is said to be invertible, if there is an element  $b \in \mathfrak{A}$  such that  $a \star b = b \star a = e$ .*

**Remark 2.15.** *The inverse  $b$  of  $a \in \mathfrak{A}$  is unique (if it exists) since  $b = b \star (a \star b') = (b \star a) \star b' = b'$ .*

**Definition 2.16** (spectrum). *Let  $\mathfrak{A}$  be a unital Banachalgebra and  $a \in \mathfrak{A}$ . The set*

$$\sigma(a) := \{ \lambda \in \mathbb{C} \mid \lambda e - a \text{ is not invertible} \}$$

*is called spectrum of  $a$  and  $\rho(a) := \mathbb{C} \setminus \sigma(a)$  is the resolvent set.*

We say that  $a \in \mathfrak{A}$  in a  $*$ -algebra  $\mathfrak{A}$  is

- *self-adjoint* if  $a^* = a$ ;
- *normal* if  $a^* \star a = a \star a^*$ ;

## 2.2 Dynamical systems and the (reduced) $C^*$ -algebra

In this section, we introduce the reduced  $C^*$ -algebra associated with a dynamical system. There are different ways to define this  $C^*$ -algebra. We follow the construction in the more general setting of groupoid  $C^*$ -algebra. The classical references for groupoid  $C^*$ -algebras are [Ren87, Ren91]. The reader is additionally referred to [Bec16, Section 3.4, Section 3.5] and references therein. The groupoid structure associated with a dynamical system is called transformation group groupoid.

A group  $G$  is a set equipped with a composition  $\circ : G \times G \rightarrow G$ , an inverse  $^{-1} : G \rightarrow G$  and a unit  $e \in G$  such that

- $\circ$  is associative;
- $g \circ g^{-1} = g^{-1} \circ g = e$  for all  $g \in G$ .

If, additionally,  $G$  is equipped with a topology,  $G$  is a topological group if the composition and the inverse are continuous.

**Definition 2.17.** *The tuple  $(X, G)$  is called a (discrete) dynamical system if  $X$  is a compact second countable Hausdorff space,  $G$  is a countable group (equipped with discrete topology) and there is a continuous  $\alpha : G \times X \rightarrow X$  satisfying  $\alpha(e, x) = x$  and  $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$  for all  $x \in X$  and  $g, h \in G$ .*

Specifically, the group  $G$  acts on  $X$  by homeomorphism. For simplification, we write  $gx$  for  $\alpha(g, x)$ . In the following, we define a  $C^*$ -algebra associated with a dynamical system  $(X, G)$ . A subset  $Y \subseteq X$  is called *invariant* if  $gY := \{gy \mid y \in Y\} \subseteq Y$  for all  $g \in G$ .

Every countable group admits a unique Haar measure  $\lambda$  given by the counting measure  $\sum_{g \in G} \delta_g$ . A probability measure  $\mu$  on the Borel  $\sigma$ -algebra of  $X$  is called  *$G$ -invariant* if  $\mu(gF) = \mu(F)$  for all Borel measurable sets  $F \subseteq X$  and  $g \in G$ . Furthermore,  $\mu$  is said to be *ergodic* if  $\mu(Y)$  is zero or one for every invariant subsets  $Y \subseteq X$ . A dynamical system  $(X, G)$  is called *uniquely ergodic* if  $(X, G)$  admits exactly one  $G$ -invariant measure on  $X$ .

**Remark 2.18.** *A fundamental statement within the theory of dynamical systems is that the space of invariant probability measure (equipped with the weak- $*$  topology) is a convex, closed subset. Its extreme point are given by the ergodic measure. Thus, this set is the closed convex hull of the ergodic measures (Krein-Milman theorem). Note that the existence of invariant measures is an assumption.*

Consider  $\mathcal{C}_c(X \times G)$  the set of continuous functions  $a : X \times G \rightarrow \mathbb{C}$  with compact support. This space is usually equipped with the inductive limit topology (i.e., a net  $(a_\iota)_\iota$  converges to  $a \in \mathcal{C}_c(X \times G)$  if there is a compact  $K \subseteq X \times G$  and an  $\iota_0$  such that  $\text{supp}(a_\iota) \subseteq K$  for all  $\iota \geq \iota_0$  and  $(a_\iota)$  converges uniformly on  $K$  to  $a$ ).

The set  $\mathcal{C}_c(X \times G)$  gets a  $*$ -algebra if equipped with the following convolution and involution:

$$(a \star b)(x, g) := \sum_{h \in G} a(x, h) b(h^{-1}x, h^{-1}g), \quad (\text{convolution})$$

$$a^*(x, g) := \overline{a(g^{-1}x, g^{-1}h)}. \quad (\text{involution})$$

We can make the  $*$ -algebra  $\mathcal{C}_c(X \times G)$  to a  $C^*$ -algebra by representing its elements as suitable operators on the Hilbert space  $\ell^2(G)$  and completing the space with the induced norm. More precisely, for  $x \in X$ , the *left-regular representation*  $\pi^x : \mathcal{C}_c(X \times G) \rightarrow \mathcal{L}(\ell^2(G))$  is defined by

$$(\pi^x(a)\psi)(g) := \sum_{h \in G} a(g^{-1}x, g^{-1}h) \psi(h), \quad \psi \in \ell^2(G), g \in G,$$

for  $a \in \mathcal{C}_c(X \times G)$ .

**Proposition 2.19.** *The family  $(\pi^x)_{x \in X}$  defines a faithful  $*$ -representation. In particular,  $\pi^x(a)$  is a linear, bounded operator on  $\ell^2(G)$  for each  $a \in \mathcal{C}_c(X \times G)$  and  $x \in X$ . Furthermore, the operator norm  $\|\pi^x(a)\|$  is bounded by*

$$C(a) := \#\{g \in G \mid \exists x \in X \text{ s.t. } a(x, g) \neq 0\} \cdot \|a\|_\infty.$$

**Proof.** You can find the proof in the more general case of groupoids in [Ren80, Proposition II.1.1, II.1.4, II.1.9], see also [Bec16, Section 3.4, Section 3.5]. Let us give here a simplified direct proof. The statement  $\pi^x$  is a  $*$ -representation follows immediately by simple algebraic computations. Note that all involved sums are finite since  $a \in \mathcal{C}_c(X \times G)$  has finite support in  $G$ . A short computation invoking Cauchy-Schwarz inequality

$$\begin{aligned} \|\pi^x(a)\psi\|^2 &= \sum_{\tilde{g} \in G} |(\pi^x(a)\psi)(\tilde{g})|^2 \\ &\leq \sum_{\tilde{g} \in G} \left( \sum_{h \in G} |a(\tilde{g}^{-1}x, \tilde{g}^{-1}h)|^{\frac{1}{2}} \cdot |a(\tilde{g}^{-1}x, \tilde{g}^{-1}h)|^{\frac{1}{2}} |\psi(h)| \right)^2 \\ &\leq \sum_{\tilde{g} \in G} \left( \sum_{h \in G} |a(\tilde{g}^{-1}x, \tilde{g}^{-1}h)| \right) \left( \sum_{h \in G} |a(\tilde{g}^{-1}x, \tilde{g}^{-1}h)| |\psi(h)|^2 \right) \\ &\leq C(a) \cdot \left( \sum_{h \in G} \left( \sum_{\tilde{g} \in G} |a(\tilde{g}^{-1}x, \tilde{g}^{-1}h)| \right) |\psi(h)|^2 \right) \\ &\leq C(a)^2 \|\psi\|^2 \end{aligned}$$

shows that  $\pi^x(a)$  is a bounded operator on  $\ell^2(G)$ . Finally it is left to show that the family of representations  $(\pi^x)_{x \in X}$  is faithful. Clearly, if  $a = 0$  then  $\pi^x(a) = 0$  for all  $x \in X$ . For the converse direction, suppose  $\pi^x(a) = 0$  for all  $x \in X$  and assume that  $a \neq 0$  which we lead as follows to a contradiction: By assumption, there is a tuple  $(y, g) \in X \times G$  such that  $a(y, g) \neq 0$ . Thus,

$$(\pi^y(a)\delta_g)(\tilde{g}) = \sum_{h \in G} a(\tilde{g}^{-1}y, \tilde{g}^{-1}h) \delta_g(h) = a(\tilde{g}^{-1}y, \tilde{g}^{-1}g) \stackrel{\tilde{g}=e}{=} a(y, g) \neq 0$$

follows for  $\tilde{g} = e$ . Hence,

$$\|\pi^y(a)\| \geq \|\pi^y(a)\delta(g)\| = \sqrt{\sum_{\tilde{g} \in G} |(\pi^y(a)\delta_g)(\tilde{g})|^2} \geq |(\pi^y(a)\delta_g)(e)| = |a(y, g)| > 0$$

is derived, a contradiction to  $\pi^x(a) = 0$  for all  $x \in X$ .  $\square$

**Remark 2.20.** *It is worth mentioning that  $\#\{g \in G \mid \exists x \in X \text{ s.t. } a(x, g) \neq 0\}$  is finite for  $a \in \mathcal{C}_c(X \times G)$  since  $G$  is countable and  $a$  has compact support.*

With this at hand, we define a norm on  $\mathcal{C}_c(X \times G)$  by  $\|a\| := \|a\|_{red} := \sup_{x \in X} \|\pi^x(a)\|$ . This norm is also called *reduced norm*.

**Proposition 2.21.** *The  $*$ -algebra  $\mathcal{C}_c(X \times G)$  equipped with the norm  $\|a\| := \sup_{x \in X} \|\pi^x(a)\|$  is a normed unital  $*$ -algebra with unit  $I := \chi_X \times \delta_e$ . Furthermore, the estimate  $\|a\|^2 \leq \|a^* \star a\|$  holds.*

**Proof.** A short computation yields

$$\|a \star b\| = \sup_{x \in X} \|\pi^x(a \star b)\| = \sup_{x \in X} \|\pi^x(a) \pi^x(b)\| \leq \sup_{x \in X} \|\pi^x(a)\| \|\pi^x(b)\| \leq \|a\| \|b\|,$$

implying that  $(\mathcal{C}_c(X \times G), \|\cdot\|)$  is a normed algebra. Similarly, the estimate  $\|a\|^2 \leq \|a^* \star a\|$  follows as it holds for the operator norm on  $\mathcal{L}(\ell^2(G))$ . For  $a \in \mathcal{C}_c(X \times G)$ , a short computation leads to  $a \star I = I \star a = a$ . Hence,  $I \in \mathcal{C}_c(X \times G)$  is the unit.  $\square$

**Corollary 2.22.** *The completion  $\mathcal{C}_{red}^*(X \rtimes G)$  of  $\mathcal{C}_c(X \times G)$  with respect to the reduced norm is a  $C^*$ -algebra.*

We call  $\mathcal{C}_{red}^*(X \rtimes G)$  the *reduced  $C^*$ -algebra of the dynamical system  $(X, G)$* .

**Remark 2.23.** *If  $G$  is not discrete or  $X$  is not compact, then  $\mathcal{C}_{red}^*(X \rtimes G)$  has not a unit. (this is actually a characterization for the reduced  $C^*$ -algebra to be unital.)*

The previous considerations lead us to the theory of random operator families. Let  $(X, G)$  be a discrete dynamical system. Every element  $a \in \mathcal{C}_{red}^*(X \rtimes G)$  induces an operator family  $A := (A_x)_{x \in X}$  with  $A_x := \pi^x(a)$  (convolution operator or integral operator with kernel  $a$ ).

**Proposition 2.24.** *Let  $(X, G)$  be a discrete dynamical system. Consider an operator family  $A := (A_x)_{x \in X}$  with  $A_x := \pi^x(a)$  induced by a normal element  $a \in \mathcal{C}_{red}^*(X \rtimes G)$ . Then the following assertions hold.*

(a) *The spectrum  $\sigma(a)$  is equal to the union  $\overline{\bigcup_{x \in X} \sigma(A_x)}$ .*

(b) *The family of operators is equivariant/covariant, i.e., the equation*

$$A_{h \cdot x} = U_h A_x U_h^{-1}$$

*holds for all  $h \in G$  where  $U_h : \ell^2(G) \rightarrow \ell^2(G)$ ,  $U_h \psi(g) := \psi(h^{-1}g)$  is unitary.*

(c) *The map  $X \ni x \mapsto A_x$  is strongly continuous on  $\mathcal{L}(\ell^2(G))$ , i.e., the limit*

$$\lim_{y \rightarrow x} \|(A_y - A_x)\psi\|$$

*is equal to zero for all  $\psi \in \ell^2(G)$  and  $x \in X$ .*

**Proof.** (a): The identity  $\sigma(a) = \overline{\bigcup_{x \in X} \sigma(A_x)}$  follows from the fact that the family of representations  $(\pi^x)_{x \in X}$  is faithful, c.f. Proposition 2.19. We refer the reader to [NP15] for a detailed discussion and just provide a short guideline: If  $\pi$  is a  $*$ -representation of a unital  $C^*$ -algebra  $\mathfrak{A}$ , then  $\sigma(\pi(a)) \subseteq \sigma(a)$  since: If  $a - \lambda \in \mathfrak{A}$  is invertible, then  $\pi(a) - \lambda$  is also invertible and so  $\rho(a) \subseteq \rho(\pi(a))$ . The converse is proven by contradiction. One shows that  $\bigoplus_{x \in X} \pi^x : \mathcal{C}_{red}^*(X \rtimes G) \rightarrow \bigoplus_{x \in X} \mathcal{L}(\ell^2(G))$  is injective, surjective (on the image) and continuous (so a  $C^*$ -isomorphism). Thus,  $\bigoplus_{x \in X} \pi^x$  preserves the spectrum. If now  $\lambda \in \sigma(a) \setminus \overline{\bigcup_{x \in X} \sigma(A_x)}$ , the inverse  $\pi^x(a - \lambda)^{-1}$  is well-defined for all  $x \in X$  and  $\|\pi^x(a - \lambda)^{-1}\|$  is uniformly (in  $X$ ) bounded as  $\lambda$  has a positive distance to  $\overline{\bigcup_{x \in X} \sigma(A_x)}$ . Consequently,  $\bigoplus_{x \in X} \pi^x(a - \lambda)$  is invertible. Since  $\bigoplus_{x \in X} \pi^x$  is an isomorphism,  $a - \lambda$  is also invertible, a contradiction.

(b): It suffices to show the identity for  $a \in \mathcal{C}_c(X \times G)$ . Then the statement follows for  $a \in \mathcal{C}_{red}^*(X \rtimes G)$  since  $\mathcal{C}_c(X \times G) \subseteq \mathcal{C}_{red}^*(X \rtimes G)$  is dense. Let  $x \in X$  and  $h \in G$ . Then, for every  $\psi \in \ell^2(G)$  and  $g \in G$ , a short computation leads to

$$\begin{aligned} (A_{hx}\psi)(g) &= \sum_{\tilde{h} \in G} a\left(g^{-1}hx \mid g^{-1}\tilde{h}\right) \cdot \psi(\tilde{h}) \\ &= \sum_{\tilde{h} \in G} a\left((h^{-1}g)^{-1}x \mid (h^{-1}g)^{-1}h^{-1}\tilde{h}\right) \cdot (U_{h^{-1}}\psi)(h^{-1}\tilde{h}) \\ &= (\pi^x(a)(U_{h^{-1}}\psi))(h^{-1}g) \\ &= (U_h A_x U_{h^{-1}}\psi)(g). \end{aligned}$$

(c): Since  $\mathcal{C}_c(X \times G) \subseteq \mathcal{C}_{red}^*(X \rtimes G)$  is a dense subset, it suffices to show the strong continuity for all normal elements of  $\mathcal{C}_c(X \times G)$ . Let  $a \in \mathcal{C}_c(X \times G)$  be normal. First, (i) it is shown that  $\|(\pi^x(a) - \pi^y(a))\psi\|$  tends to zero if  $x$  for all  $\psi \in \mathcal{C}_c(G)$ . Secondly, (ii) a  $3\varepsilon$ -argument leads to the desired strong continuity of the map  $X \ni x \mapsto \pi^x(a)$ .

(i): Consider a  $\psi \in \mathcal{C}_c(G)$ . Then, for  $x, y \in X$ , the equation

$$\|(\pi^x(a) - \pi^y(a))\psi\|^2 = \sum_{g \in G} \left| \sum_{\tilde{h} \in G} \left( a(g^{-1}x \mid g^{-1}\tilde{h}) - a(g^{-1}y \mid g^{-1}\tilde{h}) \right) \cdot \psi(\tilde{h}) \right|^2$$

holds. Since  $a \in \mathcal{C}_c(X \times G)$  and  $\psi \in \mathcal{C}_c(G)$  are compactly supported, the sums are finite. Thus, the continuity of  $a$  implies  $\lim_{y \rightarrow x} \|(\pi^x(a) - \pi^y(a))\psi\| = 0$ .

(ii): Let  $\varphi \in \ell^2(G)$  and  $\varepsilon > 0$ . Since  $\mathcal{C}_c(G) \subseteq \ell^2(G)$  is dense, there is a  $\psi \in \mathcal{C}_c(G)$  such that  $\|\psi - \varphi\| < \frac{\varepsilon}{3C}$  where  $C := \|a\| = \sup_{x \in X} \|\pi^x(a)\| < \infty$ . Then choose, by (i), an open neighborhood  $U \subseteq X$  of  $x$  such that  $\|(\pi^x(a) - \pi^y(a))\psi\| < \frac{\varepsilon}{3}$  holds for all  $y \in U$ . Consequently, the estimate

$$\|(\pi^x(a) - \pi^y(a))\varphi\| \leq \|\pi^x(a)\| \cdot \|\varphi - \psi\| + \|(\pi^x(a) - \pi^y(a))\psi\| + \|\pi^y(a)\| \cdot \|\varphi - \psi\| < \varepsilon$$

is derived for all  $y \in U$ .  $\square$

**Remark 2.25.** *If the group  $G$  is amenable (or acts amenable on  $X$ ) then the identity  $\sigma(a) = \bigcup_{x \in X} \sigma(A_x)$  holds, c.f. [Exe14, NP15].*

**Corollary 2.26.** *Let  $x \in X$  be such that  $\text{Orb}(x) := \{gx \mid g \in G\}$  is dense in  $X$ . Then  $\pi^x$  is faithful and  $\sigma(a) = \sigma(A_x)$  holds for every self-adjoint  $a \in \mathcal{C}_{red}^*(X \rtimes G)$ .*

**Proof.** Let  $\pi^x(a) = 0$ . We have to show that  $\pi^y(a) = 0$  for each  $y \in X$ . Since  $\text{Orb}(x) \subseteq X$  is dense, there is a sequence  $(g_n)$  such that  $\lim_{n \rightarrow \infty} g_n x = y$ . Since  $\pi^x(a) = 0$ ,

Proposition 2.24 (a) implies  $\pi^{g_n x}(a) = 0$ . Using Proposition 2.24 (c), we derive  $\pi^y(a) = 0$  by the strong continuity.

Let  $y \in X$ . Since the orbit  $Orb(x) \subseteq X$  is dense, there exists a sequence  $(g_n) \subseteq G$  such that  $g_n x \rightarrow y$ . Thus, by the strong continuity we get

$$\sigma(A_y) \subseteq \overline{\lim_{n \rightarrow \infty} \sigma(A_{g_n x})} := \bigcap_{n \in \mathbb{N}} \overline{\left( \bigcup_{m=n}^{\infty} \sigma(A_{g_m x}) \right)}.$$

Using the equivariance, we get  $\sigma(A_y) \subseteq \sigma(A_x)$ . Hence, we derive

$$\sigma(a) = \overline{\bigcup_{y \in X} \sigma(A_y)} \subseteq \sigma(A_x) \subseteq \sigma(A)$$

finishing the proof.  $\square$

The section is finished by characterizing the constancy of the spectrum by the minimality of the dynamical system. Recall that a dynamical system  $(X, G)$  is minimal if, for every  $x \in X$ , its orbit  $Orb(x) := \{gx \mid g \in G\} \subseteq X$  is dense.

**Proposition 2.27.** *Let  $(X, G)$  be a discrete dynamical system. Then the following assertions are equivalent.*

- (i) *The dynamical system  $(X, G)$  is minimal.*
- (ii) *For every self-adjoint  $a \in \mathcal{C}_{red}^*(X \rtimes G)$ , the spectrum  $\sigma(\pi^x(a))$  is independent of  $x \in X$ , i.e.,  $\sigma(\pi^x(a)) = \sigma(\pi^y(a))$  holds for all  $x, y \in X$ .*
- (iii) *The representation  $\pi^x$  is faithful for every  $x \in X$ .*

**Proof.** The implication (i)  $\Rightarrow$  (ii) is a well-known fact invoking Corollary 2.26, see e.g. [CFKS87, BIST89, Jit95, Len99, LS03a]. For the proof of the statement, the reader is referred to [Bec16, Theorem 3.6.9] which follows the lines of [LS03a, Theorem 4.3].  $\square$

## 2.3 Example: Hamiltonians on $\mathbb{Z}$

- let  $\mathcal{A}$  be a finite set (equipped with discrete topology)
- the elements in  $\mathcal{A}$  represent the different atomic species
- for sake of simplicity, we consider the group  $G = \mathbb{Z}$  (but all considerations hold as well for any other countable group)
- consider the space  $\mathcal{A}^{\mathbb{Z}} := \prod_{n \in \mathbb{Z}} \mathcal{A} = \{\omega : \mathbb{Z} \rightarrow \mathcal{A}\}$  equipped with the product topology (coarsest topology such that all projections  $P_n : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}$ ,  $P_n(\omega) = \omega(n)$ , are continuous)

**Remark 2.28.** *The space  $\mathcal{A}^{\mathbb{Z}}$  is our configuration space (contains all possible solids), the letters encode the different species of atoms/molecules.*

- a base for this topology is given by the sets

$$\mathcal{O}(v, N) := \{\omega \in \mathcal{A}^{\mathbb{Z}} \mid \omega|_{\{-N, \dots, N\}} = v\},$$

where  $N \in \mathbb{N}$  and  $v \in \mathcal{A}^{\{-N, \dots, N\}}$

- a sequence  $(\omega_n)$  converges to  $\omega$  if, for every  $N \in \mathbb{N}$ , there is an  $n_0 \in \mathbb{N}$  such that  $\omega_n|_{\{-N, \dots, N\}} = \omega|_{\{-N, \dots, N\}}$  for  $n \geq n_0$

- can be also defined by the following (ultra) metric

$$d(\omega, \tilde{\omega}) := \sum_{n \in \mathbb{Z}} \frac{d_0(\omega(n), \tilde{\omega}(n))}{2^{|n|}},$$

where  $d_0 : \mathcal{A} \times \mathcal{A} \rightarrow \{0, 1\}$  is the discrete metric

- $\mathcal{A}^{\mathbb{Z}}$  is compact (countable product of compact spaces) and metrizable
- $\mathbb{Z}$  acts continuously on  $\mathcal{A}^{\mathbb{Z}}$  by translation  $\alpha_n(\omega) := \omega(\cdot - n)$
- for  $\omega \in \mathcal{A}^{\mathbb{Z}}$ , denote by  $Orb(\omega) := \{\omega(\cdot - n) \mid n \in \mathbb{Z}\}$  the orbit of  $\omega$  and by  $\Omega := \overline{Orb(\omega)}$  its hull

*Example 2.29* (one-defect). Let  $\mathcal{A} := \{\aleph, \beth\}$  and consider

$$\omega(n) := \begin{cases} \aleph, & n \neq 0, \\ \beth, & n = 0, \end{cases} \in \mathcal{A}^{\mathbb{Z}}.$$

The hull  $\Omega$  of  $\omega$  equals to  $Orb(\omega) \cup \{\aleph^\infty\}$  where  $\aleph^\infty$  is the periodic two-sided infinite word having everywhere the letter  $\aleph$ . This system is uniquely ergodic and the measure is supported on  $\{\aleph^\infty\}$  which can be seen as follows:  $\omega \in \Omega$  is isolated and so every point in  $Orb(\omega)$  is isolated (since the group action is a homeomorphism on  $\mathcal{A}^{\mathbb{Z}}$ ). Suppose there is an invariant probability measure on  $\Omega$  that has support on a subset of the orbit  $Orb(\omega)$ . By the invariance of  $\mu$  and the discreteness of the elements of  $Orb(\omega)$  in  $\Omega$ , we derive  $\mu(\{\alpha_n(\omega)\}) = c > 0$  for each  $n \in \mathbb{N}$ . Then  $\sigma$ -additivity of  $\mu$  implies  $\mu(\Omega) \geq \sum_{n \in \mathbb{Z}} \mu(\{\alpha_n(\omega)\}) = \infty$ , a contradiction as  $\mu$  is a probability measure on  $\Omega$ . Thus, any invariant measure is only supported on  $\{\aleph^\infty\}$  and so it is easy to see that such an invariant probability measure is unique.

Let  $H_{\tilde{\omega}} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ ,  $\tilde{\omega} \in \Omega$ , be defined by

$$(H_{\tilde{\omega}}\psi)(n) := \psi(n-1) + \psi(n+1) + \lambda \cdot \delta_{\beth}(\tilde{\omega}(-n)) \cdot \psi(n), \quad \tilde{\omega} \in \Omega,$$

for some  $\lambda \neq 0$ . It is well-known that the spectrum  $\sigma(H_{\tilde{\omega}})$  is equal to  $[-2, 2] \cup \{E(\lambda)\}$  where  $E(\lambda) := \lambda\sqrt{1 + \frac{4}{\lambda^2}}$  for  $\tilde{\omega} \in Orb(\omega) = \Omega \setminus \{\aleph^\infty\}$ . Clearly,  $|E(\lambda)| > 2$  follows for  $\lambda \neq 0$ . Furthermore,  $\sigma(H_{\aleph^\infty}) = [-2, 2]$  holds. We will show below that this operator family arises as a self-adjoint element of the reduced  $C^*$ -algebra associated with a dynamical system  $(\Omega, \mathbb{Z})$ .

*Example 2.30*. Consider

$$\omega(n) := (\aleph \beth)^\infty = \begin{cases} \aleph, & n \text{ even} \\ \beth, & n \text{ odd} \end{cases}.$$

The hull  $\Omega$  of  $\omega$  contains exactly two elements  $(\aleph \beth)^\infty$  and  $(\beth \aleph)^\infty$  where once the origin is fixed at the letter  $\aleph$  and once at  $\beth$ . It is not difficult to see that the dynamical system  $(\Omega, \mathbb{Z})$  is uniquely ergodic and minimal. The spectrum of the operator family  $(H_{\tilde{\omega}})_{\tilde{\omega} \in \Omega}$  consists in general of 2 bands in  $\mathbb{R}$ .

Let  $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$  be closed and  $\mathbb{Z}$ -invariant. We would like to show that the Schrödinger operators  $H_\omega : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ ,  $\omega \in \Omega$ , are contained in the  $C^*$ -algebra of the dynamical system  $(\Omega, \mathbb{Z})$ . Recall that we want to consider operators of the type

$$(H_\omega\psi)(n) := \psi(n-1) + \psi(n+1) + V_\omega(n)\psi(n), \quad \omega \in \Omega,$$

where  $V_\omega(n) := p(\alpha_{-n}(\omega))$  for some  $p : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathbb{R}$  is continuous.

**Remark 2.31.** *It is worth mentioning that in the following, the element  $a \in \mathcal{C}_c(\mathcal{A}^{\mathbb{Z}} \times \mathbb{Z})$  can be interpreted as a (local) representation of the operator in terms of the "random walk on  $\mathbb{Z}$ ". Furthermore, the group  $\mathbb{Z}$  appears twice, once for the configuration space  $\mathcal{A}^{\mathbb{Z}}$  and once for the translation.*

Consider  $a := \delta_{-1} + \delta_1 + p \times \delta_0$  being an element of  $\mathcal{C}_c(\mathcal{A}^{\mathbb{Z}} \times \mathbb{Z})$ . For  $\omega \in \Omega$ , a short computation leads to

$$\begin{aligned} (\pi^\omega(a)\psi)(n) &= \sum_{k \in \mathbb{Z}} a(\alpha_{-n}(\omega), -n+k) \psi(k) \\ &= \sum_{k \in \mathbb{Z}} \left( \delta_{-1}(-n+k) + \delta_1(-n+k) + p(\alpha_{-n}(\omega)) \delta_0(-n+k) \right) \cdot \psi(k) \\ &= \psi(n-1) + \psi(n+1) + p(\alpha_{-n}(\omega)) \psi(n) \\ &= (H_\omega \psi)(n). \end{aligned}$$

Actually, we can show the following: Let  $\mathcal{R} \subseteq G$  be a finite set and consider the continuous functions  $q_h : \mathcal{A}^G \rightarrow \mathbb{C}$  and  $p : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathbb{R}$ .

$$(H_\omega \psi)(g) := \left( \sum_{h \in \mathcal{R}} q_h(g^{-1}\omega) \psi(gh^{-1}) + \overline{q_h((gh)^{-1}\omega)} \psi(gh) \right) + p(g^{-1}\omega) \psi(g), \quad \omega \in \Omega.$$

The operator family  $H_\Omega = (H_\omega)_{\omega \in \Omega}$  is called *generalized Schrödinger operator*.

**Proposition 2.32.** *For every generalized Schrödinger operator  $H_\Omega = (H_\omega)_{\omega \in \Omega}$ , there exists a self-adjoint  $a \in \mathcal{C}_c(\mathcal{A}^G \times G)$  such that  $\pi^\omega(a) = H_\omega$ . Conversely, every self-adjoint  $a \in \mathcal{C}_c(\mathcal{A}^G \times G)$  defines a generalized Schrödinger operator  $H_\Omega = (H_\omega)_{\omega \in \Omega}$  by  $H_\omega := \pi^\omega(a)$ .*

**Proof.** The proof follows the same lines as in [Bec16, Theorem 3.7.10] by replacing the pattern equivariant functions by continuous functions.  $\square$

## 2.4 The integrated density of states and the Pastur-Shubin formula

The goal of this section is to discuss/introduce an important spectral quantity for suitable operator families, called *integrated density of states (IDS)*. The latter can be interpreted as a normalized cumulative distribution function for a spectral measure associated with a bounded random operator  $A$ .

**Remark 2.33.** *Based on the non-commutative integration theory of A. Connes [Con79], [LS03a, LS03b, LPV07] introduce the integrated density of states in the setting of von Neumann algebras arising by a groupoid structure. We provide here a summary by restricting to the special case of dynamical systems. The reader is also referred to [BP17] for Delone dynamical systems of a unimodular, locally compact, second countable, Hausdorff group.*

We start with introducing the integrated density of states by a suitable trace (given by an integral over the underlying space). More precisely, we take the Pastur-Shubin formula as definition. At the end, we shortly sketch the connection with the classical approach of restricting the operator to suitable Følner sequences and defining the IDS via a limit.

The investigations on the integrated density of states goes back to the seminal works [Pas71, Shu78]. Based on this various situations have been investigated [Shu78, AS93, LPV04, Ves08, LSV11, PSS13].



### 2.4.1 A short reminder on direct integral theory

The reader is referred to [Dix81, Chapter 1] for a more detailed discussion. Let  $X$  be a Borel measurable space with positive measure  $\mu$ . For  $x \in X$ , consider a Hilbert space  $\mathcal{H}(x)$ . The family  $x \mapsto \mathcal{H}(x)$  is a *measurable field of complex Hilbert spaces* if there is an  $S \subseteq \prod_{x \in X} \mathcal{H}(x)$  such that

- (i) the map  $x \mapsto \|\psi(x)\|$  is measurable for each  $\psi \in S$ ;
- (ii) If  $\varphi \in \prod_{x \in X} \mathcal{H}(x)$  be such that for each  $\psi \in S$ , the map  $x \mapsto \langle \psi(x), \varphi(x) \rangle$  is measurable, then  $\varphi \in S$  follows;
- (iii) There is a sequence of  $\psi_j \in S, j \in \mathbb{N}$ , such that, for each  $x \in X$ , the set  $\{\psi_j(x) \mid j \in \mathbb{N}\}$  is a total set (i.e. it spans the entire Hilbert space).

Elements of  $S$  are called *measurable vector fields*. An element  $\psi \in S$  is called *square integrable* if  $\int_X \|\psi(x)\|^2 d\mu(x) < \infty$ . Denote the set of all square integrable measurable vector fields by  $\mathcal{K}$ . Then  $\mathcal{K}$  is a complex vector space. Furthermore,  $\mathcal{K}$  equipped with

$$\langle \psi, \varphi \rangle := \int_X \langle \psi(x), \varphi(x) \rangle d\mu(x)$$

defines a pre Hilbert space. Clearly,  $\psi \in \mathcal{K}$  satisfies  $\|\psi\| = 0$  if  $\psi$  vanishes  $\mu$ -almost everywhere. By identifying those elements in  $\mathcal{K}$  that are equal  $\mu$ -almost everywhere, we get a Hilbert space denoted by

$$\int_X^\oplus \mathcal{H}(x) d\mu(x).$$

This Hilbert space is called *direct integral of the  $(\mathcal{H}(x))_{x \in X}$* .

*Example 2.34.* Let  $X$  be a countable space and  $\mu := \sum_{x \in X} \delta_x$  be the counting measure on  $X$ . For each  $x \in X$ , let  $\mathcal{H}(x)$  be some Hilbert space. Equipped with the natural measurable structure (defined by the vector fields that are only supported on a finite number of  $x \in X$ ), we derive

$$\int_X^\oplus \mathcal{H}(x) d\mu(x) = \bigoplus_{x \in X} \mathcal{H}(x).$$

For  $x \in X$ , let  $A_x \in \mathcal{L}(\mathcal{H}(x))$  be a family of bounded operators. Then  $(A_x)_{x \in X}$  is called *measurable field of operators*, if, for every  $\psi \in S$ , the map  $x \mapsto A_x \psi(x)$  is measurable. Furthermore, a measurable field of operators  $(A_x)_{x \in X}$  is called *bounded* if  $\sup_{x \in X} \|A_x\|$  is finite. (suffices to define essentially bounded) An operator  $A := (A_x)_{x \in X}$  on  $\int_X^\oplus \mathcal{H}(x) d\mu(x)$  is called *decomposable* if it is measurable and (essentially) bounded. In this case, we write  $A = \int_X^\oplus A_x d\mu(x)$ .

### 2.4.2 The integrated density of states for random operators over a dynamical system

Consider the vector space  $S \subseteq \prod_{x \in X} \ell^2(G)$  of all measurable functions  $u : X \times G \rightarrow \mathbb{C}$  satisfying  $u(x, \cdot) \in \ell^2(G)$ . With the measurable structure  $S$  at hand, the family  $(\ell^2(G))_{x \in X}$  defines a measurable field of Hilbert spaces in terms of [Dix81, Part II, Chapter I.3, Def. 1], see also [LS03a, Chapter 2]. With this at hand, the direct integral space  $\int_X^\oplus \ell^2(G) d\mu(x)$  can be defined.

A family of operators  $A := (A_x)_{x \in X}$  is called *equivariant* if  $A_{gx} = T_g A_x T_g^*$  where  $T_g : \ell^2(G) \rightarrow \ell^2(G), T_g u := u(g^{-1} \cdot)$ . A measurable, bounded, equivariant family of operators  $A := (A_x)_{x \in X}$  is called *random operator*.

**Definition 2.35.** Let  $(X, G)$  be a dynamical system and  $\mu$  be a  $G$ -invariant probability measure on  $X$ . Then the associated von-Neumann algebra is defined by

$$\mathcal{N}(X, G, \mu) := \{A = (A_x)_{x \in X} \mid A \text{ measurable, bounded, equivariant}\} / \sim$$

where two random operators  $A$  and  $B$  are equivalent ( $A \sim B$ ) if  $A_x = B_x$  holds  $\mu$ -almost everywhere.

**Remark 2.36.** A von-Neumann algebra is a special type of  $C^*$ -algebra (it is defined by a  $*$ -subalgebra of a Hilbert space  $\mathcal{H}$  that is closed in the weak operator topology and contains the identity).

Every element  $A \in \mathcal{N}(X, G, \mu)$  is decomposable, i.e.,  $A = \int_X^\oplus A_x d\mu(x)$  defines an operator on the Hilbert space  $\int_X^\oplus \ell^2(G) d\mu(x)$  (with constant fibers).

**Proposition 2.37.** For every  $a \in \mathcal{C}_{red}^*(X \rtimes G)$ , the induced operator family  $A = (\pi^x(a))_{x \in X}$  defines an element in  $\mathcal{N}(X, G, \mu)$ .

**Proof.** That  $A$  is measurable follows easily for elements  $\mathcal{C}_c(X \times G)$ . Specifically, the continuity of the kernel implies that this operators are measurable. Due to denseness of  $\mathcal{C}_c(X \times G)$  in  $\mathcal{C}_{red}^*(X \rtimes G)$ , we derive that all elements of the  $C^*$ -algebra are measurable. The equivariance is proven in Proposition 2.24. Finally,  $\sup_{x \in X} \|\pi^x(a)\|$  is finite by definition of the reduced norm on  $\mathcal{C}_{red}^*(X \rtimes G)$ . Hence,  $A$  is bounded.  $\square$

**Remark 2.38.** In general, it can be that two different  $a, b \in \mathcal{C}_{red}^*(X \rtimes G)$  give rise to the same element in  $\mathcal{N}(X, G, \mu)$ . This depends on the support of the measure  $\mu$ .

**Theorem 2.39.** Let  $(X, G, \mu)$  be an ergodic dynamical system and  $A = \int_X^\oplus A_x d\mu(x) \in \mathcal{N}(X, G, \mu)$  be self-adjoint. Then there exists a  $Y \subseteq X$  of full  $\mu$ -measure and  $\Sigma, \Sigma_\bullet \subseteq \mathbb{R}$  such that

$$\sigma(A_y) = \Sigma, \quad \sigma_\bullet(A_y) = \Sigma_\bullet, \quad \text{for all } y \in Y,$$

where  $\bullet \in \{\text{disc, ess, ac, sc, pp}\}$  ( $\sigma_{pp}$  denotes the closure of the set of eigenvalues). Furthermore,  $\Sigma = \sigma(A)$  holds.

**Proof.** The reader is referred to [LPV07, Thm. 5.1] and references therein. Here is a short sketch of the proof: The ergodicity assumption implies that an invariant measurable map is constant. Thus, the map  $X \ni x \mapsto \text{tr} E_{A_x}(B)$  is constant where  $B \subseteq \mathbb{R}$  is Borel measurable and  $E_{A_x}$  is the spectral family of  $A_x$ . Since the support of the spectral family of  $A_x$  is  $\sigma(A_x)$ , the existence of a  $\mu$ -measurable subset  $Y \subseteq X$  with full measure and the existence of a subset  $\Sigma \subseteq \mathbb{R}$  satisfying  $\Sigma = \sigma(A_y)$  for all  $y \in Y$  follows since the trace is faithful. Similarly, one shows the constancy of the essential spectrum and hence, of the discrete spectrum.  $\square$

**Definition 2.40.** Let  $\mathfrak{C}$  be a unital  $C^*$ -algebra. A linear functional  $\eta$  on  $\mathfrak{C}$  is called positive if  $\eta(a^* \star a) \geq 0$  for all  $a \in A$ . A trace on  $\mathfrak{C}$  is a positive linear functional  $\tau : \mathfrak{C} \rightarrow \mathbb{C}$  with  $\tau(1) = 1$  and  $\tau(a \star b) = \tau(b \star a)$  for all  $a, b \in \mathfrak{C}$ . The trace is called faithful if  $\tau(a^* \star a) = 0$  holds only if  $a = 0$ .

**Theorem 2.41.** The map  $\tau : \mathcal{N}(X, G, \mu)^+ \rightarrow [0, \infty)$  defined by

$$\tau(A) := \int_\Omega \langle \delta_e, A_x \delta_e \rangle d\mu(x),$$

is faithful trace on  $\mathcal{N}(X, G, \mu)^+$ .

**Proof.** The reader is referred to [LPV07, Thm. 4.2] for a proof. Actually, one shows that  $\tau : \mathcal{N}(X, G, \mu)^+ \rightarrow \mathbb{C}$  is a weight, i.e.,  $\tau : \mathcal{N}(X, G, \mu)^+ \rightarrow [0, \infty)$  and it is linear. The trace property follows the same lines as in [LPS16] and [BP17]. Specifically, one shows that the unit is a Carleman operator implying that every element in the  $\mathcal{N}(X, G, \mu)$  is a Carleman operator [LPV07, Prop. 4]. Hence, the trace property follows by [LPV07, Thm. 4.2].  $\square$

**Remark 2.42.** *The trace is first only defined on the positive elements but can be uniquely extended to the whole von-Neumann algebra by linearity.*

**Definition 2.43.** *Let  $A \in \mathcal{N}(X, G, \mu)$  be self-adjoint. Then the measure  $\eta^A$  defined by  $\eta^A(B) := \tau(\chi_B(A))$  for a Borel measurable set  $B \subseteq \mathbb{R}$  is called abstract density of states.*

From now on, we always assume that  $\mu$  is an ergodic measure on  $(X, G)$ .

Let  $A$  be a self-adjoint operator with spectral family  $E_A$ . A measure  $\nu$  on  $\mathbb{R}$  is called *spectral measure*, if for every Borel measurable  $F \subseteq \mathbb{R}$ ,  $\nu(B) = 0$  if and only if  $E_A(B) = 0$ . This way,  $\eta^A$  gives rise to a spectral measure of  $A = \int_X^\oplus A_x d\mu(x)$  on  $\mathbb{R}$ , called the (*abstract*) *integrated density of states (IDS)* as shown in the following proposition. The above formula is called *Pastur-Shubin trace formula* [Pas71, Shu79].

**Proposition 2.44.** *Let  $A \in \mathcal{N}(X, G, \mu)$  be self-adjoint. The topological support*

$$\text{supp}(\eta^A) := \{ \lambda \in \mathbb{R} \mid \eta^A([\lambda - \varepsilon, \lambda + \varepsilon]) > 0 \text{ for all } \varepsilon > 0 \}$$

*coincides with the spectrum  $\sigma(A)$  of  $A$ . Due to ergodicity, the identity  $\text{supp}(\eta^A) = \sigma(A_x)$  holds for almost every  $x \in X$  (w.r.t.  $\mu$ ).*

**Proof.** The reader is referred to [LPV07] for a more detailed discussion. As discussed in the proof of Theorem 2.41,  $\tau$  is a normal weight. Hence,  $\eta^A$  is a measure. Furthermore, the faithful property implies that  $\eta^A$  is a spectral measure. With this at hand, the identity  $\text{supp}(\eta^A) = \sigma(A)$  follows by the spectral theorem. The second equality  $\text{supp}(\eta^A) = \sigma(A_x)$  follows then by Theorem 2.39.  $\square$

**Remark 2.45.** *In general,  $\eta^A$  is not a spectral measure of  $A_x$ , c.f. [AS83] and [LPV07, Remark 5.5]. It can also happen that the set of  $x \in X$  for which  $\eta^A$  is a spectral measure is zero. However, in many cases  $\eta^A$  is supported on the spectrum of a subset  $Y \subseteq X$  with full  $\mu$ -measure, c.f. Theorem 2.39.*

*One can show under suitable assumptions (see [LPV07, Cor. 5.9] for details) that for  $\mu$ -a.e.  $x \in X$ , the operator  $A_x$  has no discrete spectrum.*

Recall Example 2.29 where  $\Omega$  was given by the orbit closure of

$$\omega(n) := \begin{cases} \aleph, & n \neq 0 \\ \beth, & n = 0 \end{cases} \in \{\aleph, \beth\}^{\mathbb{Z}}$$

which we called one-defect. We have discussed that the one-defect is a uniquely ergodic dynamical system  $(\Omega, \mathbb{Z})$ . The hull  $\Omega$  was given by  $\text{Orb}(\omega) \cup \{\aleph^\infty\}$ . Furthermore, we have shown that the invariant measure  $\mu$  is only supported on  $\{\aleph^\infty\}$ .

We were considering the operator family  $H_{\tilde{\omega}} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  defined by

$$(H_{\tilde{\omega}}\psi)(n) := \psi(n-1) + \psi(n+1) + \lambda \cdot \delta_{\beth}(\tilde{\omega}(-n)) \cdot \psi(n), \quad \tilde{\omega} \in \Omega,$$

for some  $\lambda \neq 0$ . For  $\tilde{\omega} = \aleph^\infty$  we have  $\sigma(H_{\tilde{\omega}}) = [-2, 2]$  while for all other elements in  $\Omega$  the spectrum was given by  $[-2, 2] \cup \{E(\lambda)\}$ . Since the invariant measure is only supported on  $\{\aleph^\infty\}$ , we get

$$H = \int_{\Omega}^\oplus H_{\tilde{\omega}} d\mu(\tilde{\omega}) \simeq H_{\aleph^\infty}.$$

Hence,  $\Sigma = [-2, 2] = \Sigma_{ess}$  follows. Consequently, we don't have a spectral gap but for a dense set  $\Omega$  of  $\mu$ -measure zero we actually admit one spectral gap. Here,  $\Omega$  equals to  $Orb(\omega)$ .

### 2.4.3 The classical approach

For most applications, one assumes the dynamical systems to be ergodic. In this situation, for amenable groups, one finds a subset  $Y \subseteq X$  of full  $\mu$ -measure such that one can approximate  $\eta^A$  via the spectral distribution function of finite rank analogues of  $y \in Y$ . Precisely, by taking a suitable Følner  $(B_n)$  sequence in  $G$ , one defines the canonical restrictions  $A_y^n$  of  $A_y$  on  $\ell^2(B_n)$ . Clearly, the operators  $A_y^n$  are of finite rank and we can define empirical spectral distributions by counting occurrence frequencies of eigenvalues below some fixed energy level  $E \in \mathbb{R}$ . This approach has been widely studied, see e.g. [Len02, LS05, GLV07, LMV08, Ele08, Ves08, LV09, LSV11, PSS13, Pog13, Pog14, PS16a, Pog16].

Let us have a short look on the case of  $G = \mathbb{Z}$  for simplicity. Consider the sequence  $F_N := \{-N, \dots, N\}$  which defines a (tempered) Følner sequence of  $\mathbb{Z}$ . (The word tempered is a technical condition on the growth of the Følner sequence that is needed in general. Note that every Følner sequence admits a tempered Følner subsequence.)

Let  $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$  be closed and  $\mathbb{Z}$ -invariant with  $\mathbb{Z}$ -invariant ergodic measure  $\mu$  on  $\Omega$ . Consider the operators  $H_\omega : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  defined by

$$(H_\omega \psi)(n) := \psi(n-1) + \psi(n+1) + V_\omega(n)\psi(n), \quad \omega \in \Omega,$$

where  $V_\omega(n) := p(\alpha_{-n}(\omega))$  for some  $p : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathbb{R}$  is continuous. In particular, we have a random operator  $H = (H_\omega)_{\omega \in \Omega}$ . The operator  $H_\omega$  can be viewed as acting on the Cayley graph of  $\mathbb{Z}$  with generator  $\{1\}$ . Then the restriction to  $F_N$  is defined by the restrictive action on the subgraph. Specifically, we have

$$(H_\omega^{F_N} \psi)(n) = \begin{cases} \psi(-N+1) + V(-N)\psi(-N), & n = -N, \\ \psi(n-1) + \psi(n+1) + V(n)\psi(n), & -N < n < N, \\ \psi(N-1) + V(N)\psi(N), & n = N. \end{cases}$$

Consider the eigenvalue counting function

$$N_\omega^{F_N}(E) := \#\{\lambda \in \mathbb{R} \mid \lambda \text{ eigenvalue of } H_\omega^{F_N} \text{ and } \lambda \leq E\}$$

being monotone increasing and right continuous. Define

$$n_\omega^{F_N}(E) := \frac{N_\omega^{F_N}(E)}{|F_N|}$$

With this at hand, we get the following.

There exists a monotone increasing and right continuous (i.e. a distribution function)  $n : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\lim_{N \rightarrow \infty} \|n - n_\omega^{F_N}\|_\infty = 0$  for  $\mu$ -a.e.  $\omega \in \Omega$  (so in particular pointwise for  $\mu$ -a.e.  $\omega \in \Omega$ ). Furthermore,

$$n(E) = \int_\Omega \text{Tr}(\mathbf{1}_{(-\infty, E]}(H_\omega)) d\mu(\omega) = \int_\Omega \langle \delta_0, \mathbf{1}_{(-\infty, E]}(H_\omega) \delta_0 \rangle d\mu(\omega)$$

holds for  $E \in \mathbb{R}$ . Let  $\eta^H$  be the associated (unique) measure on the Borel  $\sigma$ -algebra of  $\mathbb{R}$  with distribution function  $n$ , i.e.,  $n(E) = \eta^H((-\infty, E])$ . The proof of such a theorem is

based on an almost-additive ergodic theorem. Specifically,  $N_\omega^{F_N}$  is in the following sense subadditive

$$\sup_{E \in \mathbb{R}} |N_\omega^{F \sqcup K}(E) - N_\omega^F(E) - N_\omega^K(E)| \leq b(F) + b(K)$$

where  $b$  is an error term (boundary error) satisfying  $\lim_{N \rightarrow \infty} \frac{b(F_N)}{|F_N|} = 0$ .

This method for operators on Delone sets modeling a quasicrystal structure was first applied in [LS05]. For combinatorial Cayley graphs, convergence of almost additive functions have been used in [LMV08, LSV11, LSV12]. Such a subadditive ergodic theorem for all amenable groups has been proven in [PS16b]. For the case  $\mathbb{Z}^d$ , the reader is referred to [GLV07] and to [PSS13] for general Cayley graphs of amenable countable groups.

### 3 K-theory

A label of the spectral gap  $\{g\}$  is given by the value  $\eta^A((-\infty, E])$  for some  $E \in g$  (the value of the integrated density of states assigned with the spectral gap). We know that unitary transformations do not change the spectrum. Hence, such transformations does not change the spectral gaps. Consequently, it is sufficient to label the gaps by means of the equivalence class of a projection (under unitary equivalence).

The aim is to represent the projections as suitable "integral kernels" in order to determine all possible gap labels. In light of this, (I) we study equivalence classes of projections, (II) define a semigroup structure on this set via direct sums (need to enlarge the space) and (III) we get a group by the Grothendiek construction. This naturally leads us to the K-theory (specifically, to the  $K_0$ -group). This additional structure enables us to "compute" this  $K_0$  group and hence, determine all possible gap labels.

For further background on K-theory, the reader is referred to [Bla86, MS07] as well as [BBG92, Bel92].

#### 3.1 A monoid structure on the equivalence classes of projections

Let  $\mathfrak{C}$  be a unital  $C^*$ -algebra with unit  $I$ . An element  $p \in \mathfrak{C}$  is called *projection* if  $p = p^2 = p^*$ . There are different ways to define equivalence classes on projections:

- (A)  $p \sim_a q$  if there are  $a, b \in \mathfrak{C}$  with  $p = ab$  and  $q = ba$ . (algebraic equivalence)
- (S)  $p \sim_s q$  if there is an invertible  $a \in \mathfrak{C}$  with  $apa^{-1} = q$ . (similarity)
- (H)  $p \sim_h q$  if there is a norm continuous path of projections in  $\mathfrak{C}$  from  $p$  to  $q$ , i.e.  $P : [0, 1] \rightarrow \mathfrak{C}$  is norm continuous such that  $P(t)$  projection and  $P(0) = p$  and  $P(1) = q$ . (homotopy)

where  $p, q \in \mathfrak{C}$  are projections.

**Remark 3.1.** *One can define these equivalences also on idempotents (i.e.,  $p = p^2$ ) of a Banachalgebra  $\mathfrak{A}$ . However, in the case of  $C^*$ -algebras, idempotents are always similar to a projection.*

**Remark 3.2.** *Our main view point will be to consider a spectral projection. The equivalence by similarity does not change the spectrum: Let  $a, b \in \mathfrak{C}$  such that  $a = bcb^{-1}$  for some invertible  $c \in \mathfrak{C}$*

$$a - \lambda \text{ invertible} \iff c(b - \lambda)c^{-1} \text{ invertible} \iff b - \lambda \text{ invertible}.$$

All three notions define an equivalence relation on the set of projections  $Proj(\mathfrak{C})$  of  $\mathfrak{C}$ . Furthermore, there are some relations between this equivalences. For instance,

$$p \sim_s q \iff p \sim_a q \text{ and } I - p \sim_a I - q$$

However,  $p \sim_a q$  does **not** imply  $p \sim_s q$ . On the other hand, we have

$$p \sim_a q \quad \Rightarrow \quad \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_s \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \text{ in } M_2(\mathfrak{C}).$$

Similarly, one can show that  $p \sim_h q$  implies  $p \sim_s q$  but **not** the other way around. By passing again to matrices one gets

$$p \sim_s q \quad \Rightarrow \quad \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \text{ in } M_2(\mathfrak{C}).$$

In the gap labeling theorem, usually the Murray-von Neumann equivalence on  $Proj(\mathfrak{C})$  is used. Specifically,  $p, q \in Proj(\mathfrak{C})$  are *Murray-von Neumann equivalent* ( $p \approx q$ ), if there is a  $u \in \mathfrak{C}$  (partial isometry) such that  $p = u^*u$  and  $q = uu^*$

**Remark 3.3.** *It is worth mentioning that the values of the trace of two projections that are Murray-von Neumann equivalent are the same due to the trace property. Specifically,  $\tau(p) = \tau(u^*u) = \tau(uu^*) = \tau(q)$ .*

**Lemma 3.4.** *If  $p \sim_a q$ , then  $p \approx q$  holds. In particular,  $p \sim_a q$  holds if and only if  $p \approx q$ .*

**Proof.** We refer the reader to [Bla86, Prop. 4.6.4] for a proof.  $\square$

**Proposition 3.5.** *If  $p \sim_s q$  then there is a unitary  $u \in \mathfrak{C}$ , i.e.,  $u^{-1} = u^*$ , such that  $upu^* = q$ .*

**Proof.** Since  $p \sim_s q$ , there is an invertible  $a \in \mathfrak{C}$  such that  $apa^{-1} = q$ . Thus,  $ap = qa$  and  $pa^* = a^*q$  follow. Hence,  $pa^*a = a^*qa = a^*ap$  is derived. Consequently,  $p$  commutes with the positive element  $a^*a$  and so also with its square root, namely  $p(a^*a)^{\frac{1}{2}} = (a^*a)^{\frac{1}{2}}p$ . Define  $u := a(a^*a)^{-\frac{1}{2}}$ . The element  $(a^*a)^{-\frac{1}{2}}$  is self-adjoint and commutes with  $a$ . Thus  $u$  is unitary since

$$a(a^*a)^{-\frac{1}{2}}(a(a^*a)^{-\frac{1}{2}})^* = (a^*a)^{-\frac{1}{2}}aa^*(a^*a)^{-\frac{1}{2}} = aa^*(a^*a)^{-1} = I.$$

Furthermore,

$$upu^* = apa^*(a^*a)^{-1} = qaa^*(a^*a)^{-1} = q$$

holds finishing the proof.  $\square$

The set of equivalence classes (unitary equivalence) of projections of  $\mathfrak{C}$  is usually a quit small set:

*Example 3.6.* Let  $\mathcal{K}$  be the  $C^*$ -algebra of compact operators (i.e., image of closed unit ball is relatively compact) on a separable Hilbert space  $\mathcal{H}$ . Then a projection  $p$  is compact if and only if the range is finite dimensional. Hence, two projection  $p, q \in \mathcal{K}$  are unitary equivalent if and only if they have the same dimension. Consequently, up to unitary equivalence the set of projections in  $\mathcal{K}$  is given by  $\mathbb{N}$ .

The following lemma is quit helpful.

**Lemma 3.7.** *Let  $p, q \in \mathfrak{C}$  be projections such that  $\|p - q\| < 1$ . Then  $p \sim_a q$  follows. In particular, every norm-continuous path  $t \mapsto p(t)$  of projections is made of mutually equivalent projections.*

**Proof.** We refer the reader to [BBG92, Lemma 3.1.2] or [Bla86, Proposition 4.3.2] for the proof.  $\square$

Let  $\mathfrak{C}$  be a separable (underlying topological space contains a countable dense subset) unital  $C^*$ -algebra. Define the set

$$\mathcal{P}(\mathfrak{C}) := \{p \in \mathfrak{C} \text{ projection}\} / \sim_a$$

with respect to the algebraic equivalence (Murray-von Neumann equivalence). Elements are denoted by  $[p]$ .

**Proposition 3.8.** *Let  $\mathfrak{C}$  be a separable unital  $C^*$ -algebra. Then the set  $\mathcal{P}(\mathfrak{C})$  is countable.*

**Proof.** Since  $\mathfrak{C}$  is separable, there is a countable set  $(a_n) \subseteq \mathfrak{C}$  which is dense in  $\mathfrak{C}$ . Thus, for every projection  $p$  and  $\varepsilon < \frac{1}{2}$ , there is a  $n \in \mathbb{N}$  such that  $\|p - a_n\| < \varepsilon$ . There is no loss in generality in assuming that  $a_n$  is self-adjoint by considering  $\frac{a_n + a_n^*}{2}$ . Consequently,  $\sigma(a_n) \subseteq (-\varepsilon, \varepsilon) \cup (1 - \varepsilon, 1 + \varepsilon)$  and  $\sigma(a_n) \cap (1 - \varepsilon, 1 + \varepsilon) \neq \emptyset$ . Now, let  $p_n \in \mathfrak{C}$  be the eigenprojection to  $\sigma(a_n) \cap (1 - \varepsilon, 1 + \varepsilon)$ . By construction  $\|a_n - p_n\| < \varepsilon$  follows implying  $\|p - p_n\| < 1$ . According to Lemma 3.7,  $\|p - p_n\| < 1$  implies  $p \sim_a p_n$ . Altogether, there is a projection  $p_n$  associated with each  $a_n$  and for every projection  $p \in \mathfrak{C}$ , there exists an  $n \in \mathbb{N}$  such that  $p \sim_a p_n$  implying  $\mathcal{P}(\mathfrak{C}) = \{[p_n] \mid n \in \mathbb{N}\}$ . (Note that for  $n, m \in \mathbb{N}$  it might happen that  $p_n \sim_a p_m$ , in general.)  $\square$

**Definition 3.9.** *Two projections  $p, q \in \mathfrak{C}$  are orthogonal if  $pq = qp = 0$ . We write in this case  $p \perp q$ .*

Let  $p, q \in \mathfrak{C}$  be projections satisfying  $p \perp q$ . Then  $p + q$  is again a projection and it is called *direct sum of projections*  $p \oplus q$ . Furthermore,

**Proposition 3.10.** *If  $p_1 \sim_a q_1$ ,  $p_2 \sim_a q_2$ ,  $p_1 \perp p_2$  and  $q_1 \perp q_2$ , then  $p_1 + p_2 \sim_a q_1 + q_2$  holds.*

With this at hand, we can make sense of  $[p] \oplus [q]$  by setting it  $[p + q]$  whenever  $p \perp q$ . Thus, we can define the direct sum on

$$\mathcal{E} := \{([p], [q]) \mid \exists p' \in [p], q' \in [q] \text{ such that } p'q' = q'p' = 0\} \subseteq \mathcal{P}(\mathfrak{C}) \times \mathcal{P}(\mathfrak{C}).$$

Since, two orthogonal projections commute, this direct sum on  $\mathcal{E}$  is actually commutative. Furthermore, it is associative since the sum in  $\mathfrak{C}$  is associative.

**Prolem:** In general, one could think that (by using the equivalence relation) two projections  $p$  and  $q$  can be rotated such that both get orthogonal. However, there might not be enough space in  $\mathfrak{C}$  to do so. Specifically, the set  $\mathcal{E}$  can be rather small.

To overcome this difficulty, the idea is to enlarge  $\mathfrak{C}$  so that the direct sum can be defined on all tuples. Specifically, replace  $\mathfrak{C}$  by its stabilized algebra  $\mathfrak{C} \otimes \mathcal{K}$ , where  $\mathcal{K}$  is the  $C^*$ -algebra of compact operators.

The algebraic direct limit  $M_\infty(\mathfrak{C})$  of  $M_n(\mathfrak{C})$  by the embeddings

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

from  $M_n(\mathfrak{C})$  into  $M_{n+1}(\mathfrak{C})$ . More precisely,  $M_\infty(\mathfrak{C}) = \bigsqcup_{n \in \mathbb{N}} M_n(\mathfrak{C}) / \sim$  holds, where  $\tilde{a} \in M_n(\mathfrak{C})$  and  $\tilde{b} \in M_m(\mathfrak{C})$  are equivalent if  $n \leq m$  (or  $n \geq m$ ) and  $\tilde{a}$  can be embedded in  $M_m(\mathfrak{C})$  such that it agrees with  $\tilde{b}$ . One can think of  $M_\infty(\mathfrak{C})$  as infinite matrices over  $\mathfrak{C}$  with only finitely many nonzero entries in  $\mathfrak{C}$ . The set  $M_\infty(\mathfrak{C})$  needs to be equipped with a topology. As a natural candidate we define a topology with a norm induced by the  $C^*$ -norm of  $\mathfrak{C}$ . The corresponding completion defines a  $C^*$ -algebra which is denoted by  $\mathfrak{C} \otimes \mathcal{K}$ . This algebra is called the *stable algebra* being the tensor product of  $\mathfrak{C}$  with the compact operators  $\mathcal{K}$ .



**Proposition 3.11.** *Let  $p, q \in \mathfrak{C} \otimes \mathcal{K}$ , then there exist a tuple  $(p', q') \in \mathfrak{C} \otimes \mathcal{K}$  such that  $p'q' = q'p' = 0$ ,  $p \sim_a p'$  and  $q \sim_a q'$ . In particular,  $[p] \oplus [q] := [p' + q']$  is well-defined.*

**Proof.** Invoking Lemma 3.7, one shows that for a projection  $\tilde{p} \in \mathfrak{C} \otimes \mathcal{K}$ , there is a

$N \in \mathbb{N}$  and a projection  $p \in M_N(\mathfrak{C})$  such that  $\tilde{p} \sim_a \begin{pmatrix} p & 0_N & \\ 0_N & 0_N & \\ & & \ddots \end{pmatrix}$ , see for details

[BBG92, Lem. 3.1.5]. Without loss of generality, let  $p, q \in M_N(\mathfrak{C})$  be projections. Then

$\begin{pmatrix} q & 0_N & \\ 0_N & 0_N & \\ & & \ddots \end{pmatrix} \sim_a \begin{pmatrix} 0_N & 0_N & \\ 0_N & q & \\ & & \ddots \end{pmatrix}$  where  $0_N \in M_N(\mathfrak{C})$  is the zero matrix. This equivalence can be seen by setting

$$a = \begin{pmatrix} 0_N & 0_N & \\ 1_N & 0_N & \\ & & \ddots \end{pmatrix}, \quad b = \begin{pmatrix} 0_N & q & \\ 0_N & 0_N & \\ & & \ddots \end{pmatrix}.$$

With this at hand, we get that

$$p' = \begin{pmatrix} p & 0_N & \\ 0_N & 0_N & \\ & & \ddots \end{pmatrix}, \quad q' = \begin{pmatrix} 0_N & 0_N & \\ 0_N & q & \\ & & \ddots \end{pmatrix}.$$

are orthogonal projections, namely  $p'q' = q'p' = 0$ .  $\square$

Recall that a monoid is a triple  $(M, +, e)$  where  $M$  is a set with a binary operation  $+$  (which is associative) and neutral element  $e$ .

**Corollary 3.12.** *The set  $\mathcal{P}(\mathfrak{C} \otimes \mathcal{K})$  gets an abelian monoid with the direct sum  $\oplus$  and neutral element given by the equivalence class of zero projections.*

**Remark 3.13.** *On  $\mathcal{P}(\mathfrak{C} \otimes \mathcal{K})$  the equivalence classes  $\sim_a$ ,  $\sim_s$ ,  $\sim_h$  and  $\sim_u$  are all the same [Bla86].*

## 3.2 The Grothendieck group

On the one hand, the monoid contain all the informations we are interested. But it can be a very difficult object. For technical reasons it is convenient to pass to the corresponding Grothendieck group which is described below. There exists an explicit construction to get an abelian group from an abelian monoid.

**Lemma 3.14.** *Let  $(M, +, e)$  be an abelian monoid. Then the binary relation  $\sim_G$  on  $M \times M$  defined by*

$$(x_1, y_1) \sim_G (x_2, y_2) \quad :\Leftrightarrow \quad \exists z \in M \text{ such that } x_1 + y_2 + z = x_2 + y_1 + z$$

*is an equivalence relation.*

**Remark 3.15.** *Roughly speaking  $x_1 + y_2 + z = x_2 + y_1 + z$  represents  $x_1 - y_1 = x_2 - y_2$ .*

Denote by  $G(M)$  the quotient  $M \times M / \sim_G$  and elements of  $G(M)$  are denoted by  $\langle x, y \rangle$ .

**Lemma 3.16.** *The operation on  $G(M)$  given by*

$$\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle := \langle x_1 + x_2, y_1 + y_2 \rangle$$

*is well-defined. Furthermore,  $G(M)$  equipped with the inverse  $-\langle x, y \rangle := \langle y, x \rangle$  and the neutral element  $e := \langle x, x \rangle$  is an abelian group.*

**Definition 3.17.** For an abelian monoid  $(M, +)$ , we call the abelian group  $(G(M), +)$  the Grothendieck group.

**Lemma 3.18.** Let  $(M, +)$  be an abelian monoid and  $y \in M$ . The map

$$\gamma : M \rightarrow G(M), \quad x \mapsto \langle x + y, y \rangle$$

is independent of the choice of  $y$  and a homomorphism, i.e.,  $\gamma(x + z) = \gamma(x) + \gamma(z)$ . Moreover, we have

$$G(M) = \{\gamma(x) - \gamma(z) \mid x, z \in M\}.$$

*Example 3.19.* The natural number  $\mathbb{N}$  define an abelian monoid. One can show that  $G(M)$  is isomorphic to  $\mathbb{Z}$ .

An abelian monoid is said to admit the *cancellation property* if  $x + z = y + z$  implies  $x = y$ . The natural numbers  $\mathbb{N}$  satisfy this property but the following example not. This is important as it can be shown that the Grothendieck map  $\gamma$  is injective if and only if the monoid satisfy the cancellation property, see [Bla86, Section 1.3].

*Example 3.20.* Consider the monoid  $M := \mathbb{N} \cup \{\infty\}$  with usual addition on  $\mathbb{N}$  and  $x + \infty = \infty + \infty = \infty$ . Then  $(M, +)$  does not satisfy the cancellation property and one can show that  $G(M) = \{0\}$ .

Cancellation property holds for the monoid of the projections  $\mathcal{P}(\mathfrak{C} \otimes \mathcal{K})$  if and only if  $p \sim_a q$  implies  $1 - p \sim_a 1 - q$  in  $\mathfrak{C}$ , if and only if  $p \sim_a q$  implies  $p \sim_s q$  in  $\mathfrak{C}$ , c.f. [Bla86, Section 6.4]. It is worth mentioning that in both characterizations the corresponding equivalences should hold in  $\mathfrak{C}$  and not in the matrix algebra. This is also equivalent to the fact that all invertible elements in  $\mathfrak{C}$  are dense in  $\mathfrak{C}$  which is called *stable rank 1*, c.f. [Bla86, Section 6.5]. For  $G = \mathbb{Z}$  acting on a compact space  $X$ , minimality of this system implies that  $C_{red}^*(X \rtimes \mathbb{Z})$  has stable rank 1, c.f. [Put90].

### 3.3 The $K_0$ -group

**Definition 3.21** ( $K_0$ -group). Consider a unital separable  $C^*$ -algebra  $\mathfrak{C}$ . The Grothendieck group  $K_0(\mathfrak{C}) := G(\mathcal{P}(\mathfrak{C} \otimes \mathcal{K}))$  is called  $K_0$ -group.

**Proposition 3.22.** Let  $\mathfrak{C}$  be a unital separable  $C^*$ -algebra such that  $\mathcal{P}(\mathfrak{C} \otimes \mathcal{K})$  fulfills the cancellation property.

- (i) The  $K_0$ -group  $K_0(\mathfrak{C})$  is an abelian countable group and  $\mathcal{P}(\mathfrak{C} \otimes \mathcal{K})$  embeds in  $K_0(\mathfrak{C})$  via the Grothendieck map  $\gamma$ .
- (ii) Let  $\Phi$  be a map assigning to each projection in  $\mathfrak{C}$  a real value such that  $p \sim_a q$  implies  $\Phi(p) = \Phi(q)$  and  $\Phi(p \oplus q) = \Phi(p) + \Phi(q)$ . Then  $\Phi$  canonically defines a group homomorphism  $\Phi_*$  from  $K_0(\mathfrak{C})$  into  $\mathbb{R}$ .
- (iii) Each trace  $\tau$  on  $\mathfrak{C}$  uniquely defines a group homomorphism  $\tau_*$  such that if  $p \in \mathfrak{C}$  is a projection, then  $\tau(p) = \tau_*([p])$  where  $[p]$  denotes the corresponding equivalence class in  $K_0(\mathfrak{C})$  of  $p$ .

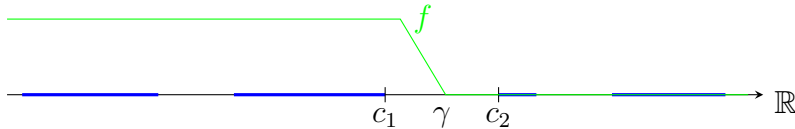
**Proof.** That the  $K_0$ -group is abelian follows from construction. That the group is countable follows by the separability of the  $C^*$ -algebra  $\mathfrak{C}$  and Proposition 3.8. The other two assertions are straightforward.  $\square$

## 4 The Gap-labeling conjecture

The theory has started with Moser [Mos81] providing examples of Schrödinger operators with nowhere dense spectrum. After this discovery, more examples were discovered. People realized that the gaps in the spectrum could be labeled such that the labels stay stable under suitable small perturbations of the Hamiltonian. Based on this, Jean Bellissard realized that the Gap labels must be of topological nature. Specifically, he connected the gap labels with the  $K$ -theory of associated  $C^*$ -algebras. For the Almost-Mathieu operator (Hofstadter butterfly), the gap labels were computed explicitly in [CEY90] being the image of the trace of the  $K_0$ -group of the rotation algebra. We also refer the reader to the review of the Gap-labeling theorem in [MS06, Appendix D] about the articles [BBG06, KP03, BOO07].

Let  $(X, G)$  be a uniquely ergodic dynamical system. Suppose that, for the reduced  $C^*$ -algebra  $\mathcal{C}_{red}^*(X \rtimes G)$ , the monoid  $\mathcal{P}(\mathcal{C}_{red}^*(X \rtimes G) \otimes \mathcal{K})$  satisfies the cancellation property.

Any self-adjoint element  $a \in \mathcal{C}_{red}^*(X \rtimes G)$  admits a compact spectrum  $\sigma(a)$  in  $\mathbb{R}$ . Its resolvent set  $\rho(a)$  is defined by  $\mathbb{R} \setminus \sigma(a)$ . A bounded connected component of  $\mathbb{R} \setminus \sigma(a)$  is called a *spectral gap*. Note that this excludes the two intervals  $(-\infty, \inf\{\sigma(a)\})$  and  $(\sup\{\sigma(a)\}, \infty)$ . Let  $\gamma$  be a spectral gap in  $\sigma(a)$ . More precisely,  $\gamma$  is an open interval  $(c_1, c_2)$ . Then the spectral projection  $\chi(a \leq E)$  for some  $E \in \gamma$  can be defined by  $f(a)$  where  $f : \mathbb{R} \rightarrow [0, 1]$  is continuous and  $f(x) = 1$  for  $x \leq c_1$  and  $f(x) = 0$  for  $x \geq c_2$ . Here  $f(a)$  is defined by the functional calculus.



In the following, we refer to a small perturbation if the eigenprojections change not too much in norm.

**Theorem 4.1.** *Let  $(X, G)$  be a uniquely ergodic dynamical system. Suppose that for the reduced  $C^*$ -algebra  $\mathcal{C}_{red}^*(X \rtimes G)$  the monoid  $\mathcal{P}(\mathcal{C}_{red}^*(X \rtimes G) \otimes \mathcal{K})$  satisfies the cancellation property.*

*Let  $a \in \mathcal{C}_{red}^*(X \rtimes G)$  be self-adjoint and  $\{\gamma\}$  be a spectral gap of  $\sigma(a) \subseteq \mathbb{R}$ .*

- (i) *The value of the IDS assigned to  $\{\gamma\}$  belongs to the countable real numbers  $[0, \tau(I)] \cap \tau_*(K_0(\mathcal{C}_{red}^*(X \rtimes G)))$ .*
- (ii) *The equivalence class  $[p(\gamma)] \in K_0(\mathcal{C}_{red}^*(X \rtimes G))$  gives a labeling of the spectral gap  $\{\gamma\}$  which is invariant under small perturbations of  $a \in \mathcal{C}_{red}^*(X \rtimes G)$  within the  $C^*$ -algebra  $\mathcal{C}_{red}^*(X \rtimes G)$ .*

**Proof.** Part (i) follows immediately from the considerations made so far. Part (ii) is a consequence of Lemma 3.7.  $\square$

Now the purpose of the Gap-labeling theory is to compute the  $K_0$ -group  $K_0(\mathcal{C}_{red}^*(X \rtimes G))$ .

Computations of such groups have been mainly analyzed for  $G = \mathbb{Z}^d$  in [BBG92, Bel92, Kel95, BKL01, KP03, BBG06, BOO03, BOO07]. In the works [BBG92, Bel92], the Gap-labeling theorem is proven for one dimensional systems ( $d = 1$ ). Moreover, several one dimensional dynamical systems are analyzed their and the corresponding Gap labels are computed such as for the Fibonacci sequence.

For the following, recall that a group action on a space  $X$  is *free* if all stabilizer groups are trivial. Specifically, there exists no  $x \in X$  and  $e \neq g \in G$  such that  $gx = x$ .

**The Gap-labeling conjecture.** *Let  $X$  be a totally disconnected compact metric space and  $X \times \mathbb{Z}^d \rightarrow X$  be a free minimal action of the group  $\mathbb{Z}^d$  on  $X$  with invariant ergodic measure  $\mu$ . Let  $\tau$  be the induced trace on  $\mathcal{C}_{red}^*(X \rtimes \mathbb{Z}^d)$ . Then*

$$\mu_*(K_0(\mathcal{C}(X))) = \tau_*(K_0(\mathcal{C}_{red}^*(X \rtimes \mathbb{Z}^d)))$$

*holds as a subset of  $\mathbb{R}$ .*

**Remark 4.2.** *A proof of this conjecture for  $d = 1$  is given in [BBG92, Bel92]. We sketch the idea respectively the ingredients of the proof in the following.*

**Definition 4.3** (Gap labels). *Elements of the image  $\tau_*(K_0(\mathcal{C}_{red}^*(X \rtimes \mathbb{Z}^d)))$  are called gap labels.*

**Remark 4.4.** *In general, solids are described by points sets  $D$  in a space  $G$ . Typically,  $G$  is a locally compact, second countable Hausdorff group and  $D \subseteq G$  is a uniformly discrete (i.e., there exists an open neighborhood  $U \ni e$  such that  $\sharp D \cap gU \leq 1$ ) and relatively compact (i.e., there exists a compact  $K \subseteq G$  such that  $\bigcup_{x \in D} xK = G$ ) subset of  $G$  that we call Delone sets. The elements of  $D$  describe the position in  $G$  of the atoms or molecules. Additionally, these points can be colored in order to describe the atomic species. In the specific case  $G = \mathbb{R}$  it is relatively easy to show that such Delone sets can be encoded by a  $\mathbb{Z}$  action on a compact space  $X$  if the Delone set is of finite local complexity. An analogous result for  $G = \mathbb{R}^d$  is much more delicate. This non-trivial problem is solved in [SW03]. More precisely, for a continuous hull  $\Omega$  of a tiling with finite local complexity in  $\mathbb{R}^d$ , the authors of [SW03] show that it is homeomorphic to  $X \times \mathbb{Z}^d$  where  $X$  is a cantor set with minimal  $\mathbb{Z}^d$  action. Based on this, [KP03] proved that  $\mathcal{C}(\Omega) \rtimes \mathbb{R}^d$  is strongly Morita equivalent to  $\mathcal{C}(X) \rtimes \mathbb{Z}^d$ . Thus, [BGR77] implies that  $(\mathcal{C}(\Omega) \rtimes \mathbb{R}^d) \otimes \mathcal{K}$  and  $(\mathcal{C}(X) \rtimes \mathbb{Z}^d) \otimes \mathcal{K}$  are isomorphic. Hence, the  $K$ -theory and their traces correspond to each other.*

Let us discuss the Gap-labeling conjecture for the case  $G = \mathbb{Z}$ . First let us point out that the  $K_0$ -group  $K_0(\mathcal{C}(X))$  of a totally disconnected space  $X$  can be computed as follows. Recall that the space of continuous functions on a compact space gets a  $C^*$ -algebra with uniform norm, c.f. Example 2.10.

**Lemma 4.5.** *Let  $(X, d)$  be a totally disconnected compact metric space. Then the abelian groups  $K_0(\mathcal{C}(X))$  and  $\mathcal{C}(X, \mathbb{Z})$  are isomorphic.*

**Proof.** Let  $p \in \mathcal{C}(X) \otimes \mathcal{K}$  be a projection which we can view as a continuous map from  $X$  into  $\mathcal{K}$ . Set  $f_p(x) := \text{Tr}(p(x))$ . Clearly,  $f_p$  takes only integer values and so  $f_p \in \mathcal{C}(X, \mathbb{N})$ . Now the claim is that the map

$$\Phi : K_0(\mathcal{C}(X)) \rightarrow \mathcal{C}(X, \mathbb{Z}), \quad p \mapsto f_p,$$

defines a group isomorphism. We show that (i)  $\Phi$  depends only on the equivalence class, (ii)  $\Phi$  is a group homomorphism, (iii)  $\Phi$  is surjective and (iv)  $\Phi$  is injective.

(i): Let  $p \sim_a q$ . Due to Lemma 3.4, there is a  $u \in \mathcal{C}(X) \otimes \mathcal{K}$  such that  $p = u^*u$  and  $q = uu^*$ . Using the trace property, we get

$$f_p(x) = \text{Tr}(u^*(x)u(x)) = \text{Tr}(u(x)u^*(x)) = f_q(x).$$

(ii): Let  $[p], [q] \in K_0(\mathcal{C}(X))$  be such that  $pq = qp = 0$ . Thus,

$$f_{p \oplus q}(x) = \text{Tr}(p(x) \oplus q(y)) = \text{Tr}(p(x)) + \text{Tr}(q(y)) = f_p(x) + f_q(y)$$

follows since the trace is linear.

(iii): Let  $f \in \mathcal{C}(X, \mathbb{Z})$  and set  $X_n := \{x \in X \mid f(x) = n\}$ ,  $n \in \mathbb{Z}$ . Then the  $X_n$ 's form a partition of clopen sets (i.e., open and closed) of  $X$ , since  $f$  is continuous and  $\{n\}$  is clopen. Denote by  $\chi_n \in \mathcal{C}(X)$  the characteristic function of  $X_n$  and  $\pi_n := 1_{|n|} \oplus 0 \in \mathcal{K}$  where  $1_{|n|}$  is an  $|n| \times |n|$  dimensional matrix. Define the projections  $p := \sum_{n \in \mathbb{N}} \pi_n \chi_n$  and  $q := \sum_{n \in \mathbb{N}} \pi_n \chi_{-n}$  in  $\mathcal{C}(X) \otimes \mathcal{K}$ . Its immediate to show that  $f = f_p - f_q$  showing that  $\Phi$  is surjective.

(iv): It suffices to consider projections in  $\mathcal{C}(X) \otimes \mathcal{K}$ . Consider the projections  $p, q \in \mathcal{C}(X) \otimes \mathcal{K}$ . The equation  $f_p - f_q = 0$  implies that  $p$  and  $q$  have the same partition  $\{X_n \mid n \in \mathbb{N}\}$ . Thus, it suffices to show  $p \approx \sum_{n \in \mathbb{N}} \pi_n \chi_n$  and  $q \approx \sum_{n \in \mathbb{N}} \pi_n \chi_n$ .

The map  $f_p : X \rightarrow \mathcal{K}$  is uniformly continuous by compactness of  $X$ . Hence, for  $0 < \varepsilon < 1$ , there is a  $\delta > 0$  such that  $\|f_p(x) - f_p(y)\| < \varepsilon$  whenever,  $d(x, y) < \delta$ . Hence, for  $n \in \mathbb{N}$ , every  $X_n$  can be partitioned in  $X_{n,k}$  such that  $x, y \in X_{n,k}$  have at most distance  $\delta$ . For any pair  $n, k$  choose some  $x_{n,k} \in X_{n,k}$ . Define  $p_\varepsilon := \sum_{n,k} p(x_{n,k}) \chi_{n,k}$ . By construction, we derive

$$\|p - p_\varepsilon\|_\infty = \sup_{x \in X} \|p(x) - p_\varepsilon(x)\| \leq \varepsilon < 1.$$

Thus, Lemma 3.7 and Lemma 3.4 imply  $p \approx p_\varepsilon$ . Furthermore,  $p(x_{n,k})$  is a fixed projection of dimension  $n$  in  $\mathcal{K}$ . Thus, there is a  $u_{n,k} \in \mathcal{K}$  such that

$$p(x_{n,k}) = u_{n,k} u_{n,k}^*, \quad \pi_n = u_{n,k}^* u_{n,k}.$$

Set  $u = \sum_{n,k} u_{n,k} \chi_{n,k} \in \mathcal{C}(X) \otimes \mathcal{K}$ . Hence,  $p_\varepsilon = uu^*$  and  $\sum_{n \in \mathbb{N}} \pi_n \chi_n = u^*u$  follow by construction implying  $\sum_{n \in \mathbb{N}} \pi_n \chi_n \approx p_\varepsilon$ . Consequently,  $p \approx \sum_{n \in \mathbb{N}} \pi_n \chi_n$  follows by the previous considerations.  $\square$

Based on this observation the Gap-labeling conjecture becomes very interesting as it would provide the opportunity to actually determine all gap labels if the measure is known. So let us discuss the conjecture for  $d = 1$ . The containment

$$\mu_*(K_0(\mathcal{C}(X))) \subseteq \tau_*(K_0(\mathcal{C}_{red}^*(X \rtimes \mathbb{Z})))$$

is the trivial direction. One uses that the  $C^*$ -algebra is a groupoid  $C^*$ -algebra. One can show that characteristic functions of clopen subsets of the groupoid  $\Gamma = X \rtimes \mathbb{Z}$  give rise to projections with trace given by their integral over the measure  $\mu$ . Since the groupoid  $C^*$ -algebra is generated by these functions, the above mentioned inclusion follows.

The hard direction is the converse inclusion

$$\mu_*(K_0(\mathcal{C}(X))) \supseteq \tau_*(K_0(\mathcal{C}_{red}^*(X \rtimes \mathbb{Z}^d))).$$

For this, the  $K_1$ -group  $K_1(\mathcal{C}_{red}^*(X \rtimes G))$  plays also an important role. The  $K_1$ -group is defined by a group structure on the connected components of invertible (infinite) matrices with values in the corresponding  $C^*$ -algebra. We refrain from giving a precise definition and refer to [Bla86, MS07]. The idea is to use certain identities between the  $K_0$ - and the  $K_1$ -group for certain crossed and tensor products. Here the Pimsner-Voiculescu Exact sequence theorem [PV80] is used in dimension  $d = 1$ .

# Bibliography

- [AS93] T. Adachi and T. Sunada, *Density of states in spectral geometry*, Comment. Math. Helv. **68** (1993), no. 3, 480–493. MR 1236765
- [AS83] J. Avron and B. Simon, *Almost periodic Schrödinger operators. II. The integrated density of states*, Duke Math. J. **50** (1983), no. 1, 369–391. MR 700145
- [BP17] S. Beckus and F. Pogorzelski, *Delone dynamical systems and spectral convergence*.
- [Bec16] Siegfried Beckus, *Spectral approximation of aperiodic Schrödinger operators*, Ph.D. thesis, 2016, Friedrich-Schiller-Universität Jena (Germany), <https://arxiv.org/abs/1610.05894>, p. 244.
- [Bel92] J. Bellissard, *Gap Labelling Theorems for Schrödinger operators*, From number theory to physics (Les Houches, 1989), Springer, Berlin, 1992, pp. 538–630. MR 1221111 (94e:46120)
- [BBG06] J. Bellissard, R. Benedetti, and J.-M. Gambaudo, *Spaces of tilings, finite telescopic approximations and gap-labeling*, Comm. Math. Phys. **261** (2006), no. 1, 1–41. MR 2193205 (2007c:46063)
- [BIST89] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, *Spectral properties of one-dimensional quasi-crystals*, Comm. Math. Phys. **125** (1989), no. 3, 527–543. MR 1022526 (90m:82043)
- [BKL01] J. Bellissard, J. Kellendonk, and A. Legrand, *Gap-labelling for three-dimensional aperiodic solids*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 6, 521–525. MR 1834062
- [BBG92] Jean Bellissard, Anton Bovier, and Jean-Michel Ghez, *Gap labelling theorems for one-dimensional discrete Schrödinger operators*, Rev. Math. Phys. **4** (1992), no. 1, 1–37. MR 1160136
- [BOO03] M.-T. Benameur and H. Oyono-Oyono, *Gap-labelling for quasi-crystals (proving a conjecture by J. Bellissard)*, Operator algebras and mathematical physics (Constanța, 2001), Theta, Bucharest, 2003, pp. 11–22. MR 2018220
- [BOO07] ———, *Index theory for quasi-crystals. I. Computation of the gap-label group*, J. Funct. Anal. **252** (2007), no. 1, 137–170. MR 2357353
- [Bla86] B. Blackadar (ed.), *K-Theory for Operator Algebras*, Mathematical Sciences Research Institute Publications, vol. 5, Springer-Verlag New York, 1986.
- [Bla17] ———, *Theory of  $c^*$ -Algebras and von Neumann Algebras*, 2017.

- [BGR77] L. G. Brown, P. Green, and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of  $C^*$ -algebras*, Pacific J. Math. **71** (1977), no. 2, 349–363. MR 0463928
- [CEY90] M. D. Choi, G. A. Elliott, and N. Yui, *Gauss polynomials and the rotation algebra*, Invent. Math. **99** (1990), no. 2, 225–246. MR 1031901
- [Con79] A. Connes, *Sur la théorie non commutative de l'intégration*, Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978), Lecture Notes in Math., vol. 725, Springer, Berlin, 1979, pp. 19–143. MR 548112 (81g:46090)
- [CFKS87] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, study ed., Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987. MR 883643 (88g:35003)
- [Dix77] J. Dixmier,  *$C^*$ -algebras*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, Translated from the French by Francis Jellet, North-Holland Mathematical Library, Vol. 15. MR 0458185 (56 #16388)
- [Dix81] ———, *von Neumann algebras*, North-Holland Mathematical Library, vol. 27, North-Holland Publishing Co., Amsterdam-New York, 1981, With a preface by E. C. Lance, Translated from the second French edition by F. Jellet. MR 641217 (83a:46004)
- [Ele08] Gábor Elek,  *$L^2$ -spectral invariants and convergent sequences of finite graphs*, J. Funct. Anal. **254** (2008), no. 10, 2667–2689.
- [Exe14] R. Exel, *Invertibility in groupoid  $C^*$ -algebras*, Operator theory, operator algebras and applications, Oper. Theory Adv. Appl., vol. 242, Birkhäuser/Springer, Basel, 2014, pp. 173–183. MR 3243289
- [GLV07] M. J. Gruber, D. H. Lenz, and I. Veselić, *Uniform existence of the integrated density of states for random Schrödinger operators on metric graphs over  $\mathbb{Z}^d$* , J. Funct. Anal. **253** (2007), no. 2, 515–533. MR 2370087
- [Jit95] S. Y. Jitomirskaya, *Almost everything about the almost Mathieu operator. II*, XIth International Congress of Mathematical Physics (Paris, 1994), Int. Press, Cambridge, MA, 1995, pp. 373–382. MR 1370694 (96m:82035)
- [KP03] J. Kaminker and I. Putnam, *A proof of the gap labeling conjecture*, Michigan Math. J. **51** (2003), no. 3, 537–546. MR 2021006
- [Kel95] J. Kellendonk, *Noncommutative geometry of tilings and gap labelling*, Rev. Math. Phys. **7** (1995), no. 7, 1133–1180. MR 1359991
- [Len99] D. Lenz, *Random operators and crossed products*, Math. Phys. Anal. Geom. **2** (1999), no. 2, 197–220. MR 1733886 (2001b:46108)
- [Len02] ———, *Singular spectrum of Lebesgue measure zero for one-dimensional quasicrystals*, Comm. Math. Phys. **227** (2002), no. 1, 119–130. MR 1903841 (2003k:47043)
- [LMV08] D. Lenz, P. Müller, and I. Veselić, *Uniform existence of the integrated density of states for models on  $\mathbb{Z}^d$* , Positivity **12** (2008), no. 4, 571–589.

- [LPV07] D. Lenz, N. Peyerimhoff, and I. Veselić, *Groupoids, von Neumann algebras and the integrated density of states*, Math. Phys. Anal. Geom. **10** (2007), no. 1, 1–41. MR 2340531
- [LPS16] D. Lenz, F. Pogorzelski, and M. Schmidt, *The ihara zeta function for infinite graphs*, 2016, arXiv:1408.3522.
- [LSV11] D. Lenz, F. Schwarzenberger, and I. Veselić, *A Banach space-valued ergodic theorem and the uniform approximation of the integrated density of states*, Geom. Dedicata **150** (2011), 1–34, Erratum in DOI: 10.1007/s10711-011-9657-1. MR 2753695
- [LS03a] D. Lenz and P. Stollmann, *Algebras of random operators associated to Delone dynamical systems*, Math. Phys. Anal. Geom. **6** (2003), no. 3, 269–290. MR 1997916 (2004j:46092)
- [LS03b] ———, *Delone dynamical systems and associated random operators*, Operator algebras and mathematical physics (Constanța, 2001), Theta, Bucharest, 2003, pp. 267–285. MR 2018235
- [LS05] ———, *An ergodic theorem for Delone dynamical systems and existence of the integrated density of states*, J. Anal. Math. **97** (2005), 1–24. MR 2274971
- [LV09] D. Lenz and I. Veselić, *Hamiltonians on discrete structures: jumps of the integrated density of states and uniform convergence*, Math. Z. **263** (2009), no. 4, 813–835.
- [LPV04] Daniel Lenz, Norbert Peyerimhoff, and Ivan Veselić, *Integrated density of states for random metrics on manifolds*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 733–752. MR 2044055
- [LSV12] Daniel Lenz, Fabian Schwarzenberger, and Ivan Veselić, *Erratum to: A Banach space-valued ergodic theorem and the uniform approximation of the integrated density of states [mr2753695]*, Geom. Dedicata **159** (2012), 411–413. MR 2944541
- [MS07] R. Matthes and W. Szymanski, *Lecture notes on the  $k$ -theory of operator algebras*, www.impan.gov.pl/Manuals/K\_theory.pdf, 2007.
- [MS06] C. C. Moore and C. L. Schochet, *Global analysis on foliated spaces*, second ed., Mathematical Sciences Research Institute Publications, vol. 9, Cambridge University Press, New York, 2006. MR 2202625
- [Mos81] J. Moser, *An example of a Schroedinger equation with almost periodic potential and nowhere dense spectrum*, Comment. Math. Helv. **56** (1981), no. 2, 198–224. MR 630951
- [Mur90] G. J. Murphy,  *$C^*$ -algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990. MR 1074574 (91m:46084)
- [NP15] V. Nistor and N. Prudhon, *Exhausting families of representations and spectra of pseudodifferential operators*, 2015, arXiv:1411.7921.
- [Pas71] L. A. Pastur, *Selfaverageability of the number of states of the Schrödinger equation with a random potential*, Mat. Fiz. i Funkcional. Anal. (1971), no. Vyp. 2, 111–116, 238.



- [PV80] M. Pimsner and D. Voiculescu, *Imbedding the irrational rotation  $C^*$ -algebra into an AF-algebra*, J. Operator Theory **4** (1980), no. 2, 201–210. MR 595412
- [Pog13] F. Pogorzelski, *Almost-additive ergodic theorems for amenable groups*, J. Funct. Anal. **265** (2013), no. 8, 1615–1666. MR 3079231
- [Pog14] ———, *Convergence theorems for graph sequences*, Internat. J. Algebra Comput. **24** (2014), no. 8, 1233–1251. MR 3296365
- [Pog16] ———, *Corrigendum and Addendum to “Almost-additive ergodic theorems for amenable groups” [J. Funct. Anal. 265 (2013) 1615–1666] [MR3079231]*, J. Funct. Anal. **271** (2016), no. 11, 3393–3401. MR 3554710
- [PS16a] F. Pogorzelski and F. Schwarzenberger, *A Banach space-valued ergodic theorem for amenable groups and applications*, J. Anal. Math. **130** (2016), 19–69. MR 3574647
- [PS16b] ———, *A Banach space-valued ergodic theorem for amenable groups and applications*, J. Anal. Math. **130** (2016), 19–69. MR 3574647
- [PSS13] F. Pogorzelski, F. Schwarzenberger, and C. Seifert, *Uniform existence of the integrated density of states on metric Cayley graphs*, Lett. Math. Phys. **103** (2013), no. 9, 1009–1028. MR 3077962
- [Put90] I. F. Putnam, *On the topological stable rank of certain transformation group  $C^*$ -algebras*, Ergodic Theory Dynam. Systems **10** (1990), no. 1, 197–207. MR 1053808
- [Ren80] J. Renault, *A groupoid approach to  $C^*$ -algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980.
- [Ren87] ———, *Représentation des produits croisés d’algèbres de groupoïdes*, J. Operator Theory **18** (1987), no. 1, 67–97. MR 912813
- [Ren91] ———, *The ideal structure of groupoid crossed product  $C^*$ -algebras*, J. Operator Theory **25** (1991), no. 1, 3–36, With an appendix by Georges Skandalis. MR 1191252
- [SW03] L. Sadun and R. F. Williams, *Tiling spaces are Cantor set fiber bundles*, Ergodic Theory Dynam. Systems **23** (2003), no. 1, 307–316. MR 1971208
- [Shu78] M.A. Shubin, *Almost periodic functions and partial differential operators*, Russian Math. Surv. **33** (1978), 1–52.
- [Shu79] M.A. Shubin, *Spectral theory and the index of elliptic operators with almost-periodic coefficients*, Russ. Math. Surveys **34** (1979), no. 2, 109–157.
- [Ves08] I. Veselić, *Existence and regularity properties of the integrated density of states of random Schrödinger operators*, Lecture Notes in Mathematics, vol. 1917, Springer-Verlag, Berlin, 2008. MR 2378428