The connection of the K-theory with the Gap-labeling theorem of Schrödinger operators

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Abstract

This notes is based on a lecture given at the Israel Institute of Technology in Haifa. The reader is invited to send comments and remarks to the author to improve this notes.

Contents

1 Introduction

1.1 Motivation - Schrödinger operators

- solid state physics: long time behavior of particle

$\Rightarrow$ (QM) study spectral theory of Schrödinger operators

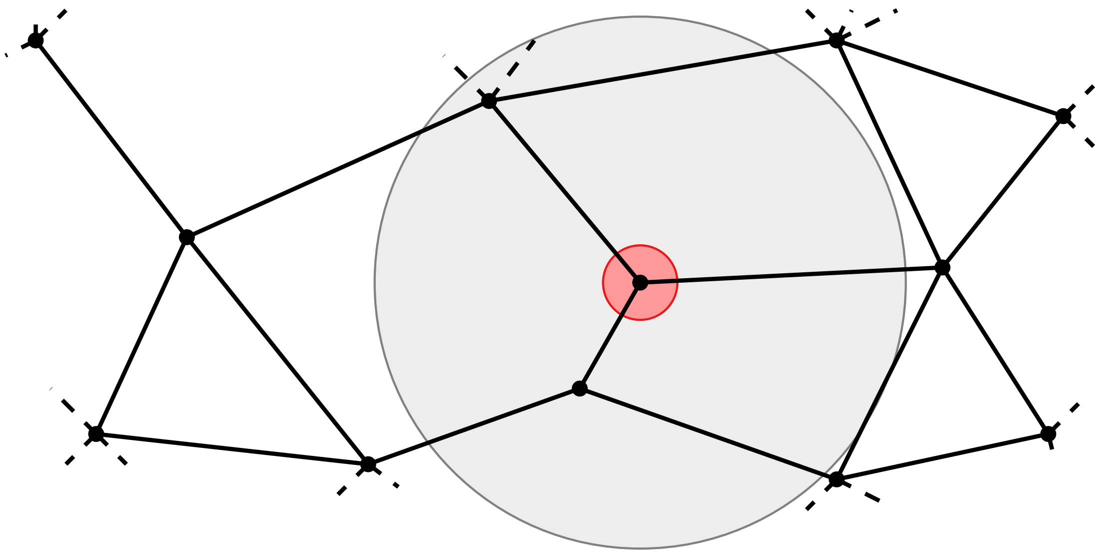
$$H := -\Delta + V \qquad \text{(self-adjoint)}$$

continuous model	discrete model
H unbounded	H bounded (depends on vertex degree)
Example on $L^2(\mathbb{R})$	Example on $\ell^2(\mathbb{Z})$
$H = -\frac{d^2}{dx^2} + V$	$\frac{d}{dx} f(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$

$$\frac{d^2}{dx^2} f(x_0) \approx \frac{\big(f(x_0+h)-f(x_0)\big)-\big(f(x_0+2h)-f(x_0+h)\big)}{h^2}$$

$$\overset{h=1}{=} -f(x_0) - f(x_0+2) + 2 \cdot f(x_0+1)$$

$$\big(H\psi\big)(n) := \psi(n-1) + \psi(n+1) + V(n)\psi(n)$$



1.2 The integrated density of states (IDS)

- the following approach has been widely analyzed as discussed later, see e.g. [Bel92, Len02, LS05, LMV08, Ele08, LV09, LSV11, Pog14, PS16]

- consider the Schrödinger operator $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$\big(H\psi\big)(n) := \psi(n-1) + \psi(n+1) + V(n)\psi(n)$$

- take an exhausting sequence $F_N := \{-N, -N+1, \dots, N\} \subseteq \mathbb{Z}, N \in \mathbb{N}$, and denote by H_{F_N} the restriction of H to F_N with Dirichlet boundary conditions
- consider

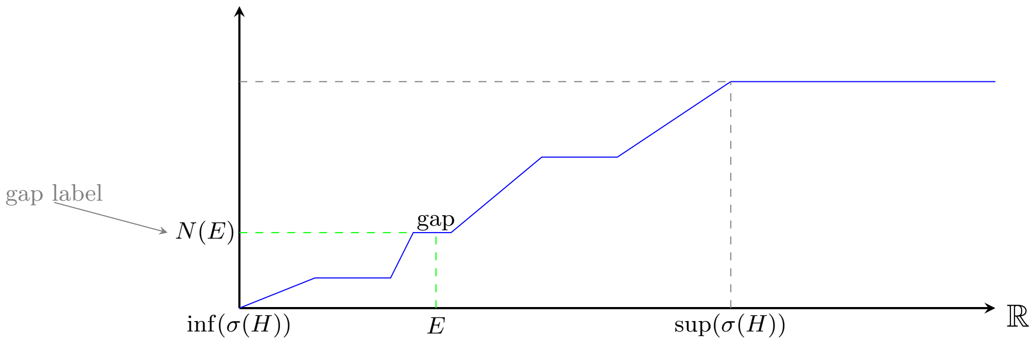
$$N_{F_N}(E) := \sharp\{\lambda \in \mathbb{R} \,|\, \lambda \text{ eigenvalue of } H_{F_N} \text{ and } \lambda \leq E\}$$

Definition 1.1. *The limit (if it exists)*

$$N(E) := \lim_{N\to\infty} \frac{N_{F_N}(E)}{\sharp F_N}$$

is called integrated density of states (IDS) of H.

- $\chi(H_{F_N} \leq E)$ is the eigenprojection onto the eigenspace of H_{F_N} with energies less or equal than E
- then $N_{F_N}(E) = tr\big(\chi(H_{F_N} \leq E)\big)$



- People discovered in examples that the spectral gaps can be labeled such that the labeling is stable under small perturbations of the Hamiltonian.
- Based on this experiences Jean Bellissard realized that the Gap labeling should be of topological nature. Thus, he connected the gap labels with the K-theory (K_0-group) and the trace of associated C^*-algebras.

Aim: Determine the gap labels of a given Schrödinger operator.

1.3 Strategy

- dynamical approach $\to$ view operators as suitable integral operators with kernels on the dynamical system (C^*-algebra approach)
- Pastur-Shubin formula: write $N(E)$ as a trace of the corresponding eigenprojections
- under suitable ergodicity assumptions the trace is given by an integral over the (unique) ergodic measure
- define a group structure on the "equivalence classes" (by unitary) of the eigenprojection (K_0-group)
- then the possible gap labels are contained in the image of the trace of the K_0 group

2 C^*-algebras associated with dynamical systems and the integrated density of states

2.1 C^*-algebras

In the following section, funcamental notions of C^*-algebras are introduced. This is just a short summary. The reader is referred to [Dix77, Dix81, Mur90, Bla17] and references therein for further background.

Definition 2.1 (algebra). *An algebra $\mathfrak{A}$ is a vector space (over $\mathbb{C}$) with multiplication $\star : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}, (a, b) \mapsto a \cdot b$, satisfying*

- $a \star (b \star c) = (a \star b) \star c$ ***(associative)***
- $(a + b) \star c = a \star c + b \star c$
  $a \star (b + c) = a \star b + a \star c$ ***(distributive)***
- $\alpha \cdot (a \star b) = (\alpha \cdot a) \star b = a \star (\alpha \cdot b)$

for all $a, b, c \in \mathfrak{A}$ and $\alpha \in \mathbb{C}$. An algebra A is called unital *if there is an $e \in \mathfrak{A}$ such that $e \star a = a \star e = a$ for all $a \in \mathfrak{A}$. Then e is called* unit.

Remark 2.2. *If $\mathfrak{A}$ is a unital algebra, then the unit e is unique. ($e = e \star e' = e'$) In general, the multiplication is not commutative. An algebra $\mathfrak{A}$ is said to be* commutative *if $a \star b = b \star a$ for all $a, b \in \mathfrak{A}$ and otherwise A is* noncommutative.

Definition 2.3 (Banachalgebra). *A tuple $(\mathfrak{A}, \|\cdot\|)$ is called a* normed algebra *if $\mathfrak{A}$ is an algebra and the map $\|\cdot\| : \mathfrak{A} \to [0, \infty)$ is a norm satisfying $\|a \star b\| \leq \|a\| \, \|b\|$ for all $a, b \in \mathfrak{A}$. If $\mathfrak{A}$ is additional unital, we require $\|e\| = 1$. Furthermore, a normed algebra $(\mathfrak{A}, \|\cdot\|)$ is called* Banachalgebra *if $(\mathfrak{A}, \|\cdot\|)$ is a complete space.*

Example 2.4. The normed space $(\ell^1(\mathbb{Z}), \|\cdot\|_1)$ with $\|\psi\|_1 := \sum_{n\in\mathbb{Z}} |\psi(n)|$ and multiplication

$$\big(\psi \star \varphi\big)(n) := \sum_{k\in\mathbb{Z}} \psi(n-k)\varphi(k)$$

is a (commutative) Banachalgebra with unit $\delta_0 \in \ell^1(\mathbb{Z})$ defined by $\delta_0(n) = 1$ if $n = 1$ and otherwise $\delta_0(n) = 0$.

Remark 2.5. *The constraint $\|a \star b\| \leq \|a\| \, \|b\|$ guarantees the continuity of the multiplication on A.*

Definition 2.6 ($*$-algebra). *Let $\mathfrak{A}$ be an algebra. A map $^* : \mathfrak{A} \to \mathfrak{A}$ is called* involution *if*

- $(a + \alpha b)^* = a^* + \overline{\alpha} b^*$
- $(a \star b)^* = b^* \star a^*$
- $(a^*)^* = a$

holds for all $a, b \in \mathfrak{A}$ and $\alpha \in \mathbb{C}$. Then $(\mathfrak{A}, *)$ is called $*$-algebra / involutive algebra.

Definition 2.7 (C^*-algebra)**.** *Let $(\mathfrak{C}, *, \|\cdot\|)$ be a $*$-Banachalgebra. Then $\mathfrak{C}$ is called a* C^*-algebra *if*

$$\|a\|^2 \leq \|a^* \star a\|\,, \qquad a \in \mathfrak{C}\,,$$

holds.

Remark 2.8. *The constraint $\|a\|^2 \leq \|a^* \star a\|$ is equivalent to $\|a\|^2 = \|a^* \star a\|$. For a C^*-algebra, $* : \mathfrak{C} \to \mathfrak{C}$ is isometric (i.e., $\|a^*\| = \|a\|$) since*

$$\|a\|^2 \leq \|a^* \star a\| \leq \|a^*\| \, \|a\| \quad \Rightarrow \quad \|a\| \leq \|a^*\|.$$

Example 2.9 (Complex plane). The set $\mathfrak{C} = \mathbb{C}$ with pointwise multiplication and involution defined by complex conjugation is a unital (commutative) C^*-algebra with unit $e = 1$ where $\|\alpha\| := |\alpha|$.

Example 2.10. Let X be a topological space (locally compact). The set $\mathfrak{C} = \mathcal{C}_0(X)$ with pointwise multiplication, uniform norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$ and involution defined by complex conjugation is a (commutative) C^*-algebra. It is unital if and only if X is a compact space.

Example 2.11 (Linear bounded operators). Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{L}(\mathcal{H})$ be the set of all linear, bounded operators $T : \mathcal{H} \to \mathcal{H}$ with multiplication defined by composition, involution defined by the adjoint of an operator and operator norm $\|T\| := \sup_{\|\psi\| \leq 1} \|T\psi\|$. Then $\mathfrak{C} := \mathcal{L}(\mathcal{H})$ is a (noncommutative) C^*-algebra. For $T \in \mathcal{L}(\mathcal{H})$, the norm closure of the set $\{p(T) \,|\, p \text{ polynomial}\}$ is a (commutative) C^*-subalgebra.

Example 2.12. The normed space $(\ell^1(\mathbb{Z}), \|\cdot\|_1)$ with $\|\psi\|_1 := \sum_{n \in \mathbb{Z}} |\psi(n)|$ and convolution

$$\big(\psi \star \varphi\big)(n) := \sum_{k \in \mathbb{Z}} \psi(n-k)\varphi(k)$$

is a (commutative) Banachalgebra with unit $\delta_0 \in \ell^1(\mathbb{Z})$ defined by $\delta_0(n) = 1$ if $n = 1$ and otherwise $\delta_0(n) = 0$. The map $* : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ defined by $\psi^*(n) := \overline{\psi(-n)}$ defines a involution. Hence, $(\ell^1(\mathbb{Z}), *, \|\cdot\|_1)$ is a $*$-Banachalgebra but not a C^*-algebra: Let $\psi \in \ell^1(\mathbb{Z})$ be defined by $\psi(0) = 1, \psi(1) = \psi(2) = -1$ and $\psi(n) = 0$ if $n \in \mathbb{Z} \setminus \{0, 1, 2\}$. Then

$$\big(\psi^* \star \psi\big)(n) = \begin{cases} 3, & n = 0, \\ -1, & n = \pm 2, \\ 0, & \text{otherwise} \end{cases}$$

holds. Thus, $\|\psi^* \star \psi\| = 5$ while $\|\psi\|^2 = 9$. Consequently, the C^*-identity is not satisfied.

Definition 2.13 (Representation)**.** *Let $\mathfrak{A}$ be an $*$-algebra. Then a pair $(\pi, \mathcal{H})$ is called a* $*$-representation of $\mathfrak{A}$ *if $\mathcal{H}$ is a Hilbert space and $\pi : \mathfrak{A} \to \mathcal{L}(\mathcal{H})$ is a $*$-homomorphism, i.e.,*

(i) π is linear;

(ii) π is multiplicative, i.e., $\pi(f \star g) = \pi(f)\pi(g)$ holds for all $f, g \in \mathfrak{A}$;

(iii) π preserves the involution, i.e., $\pi(f^*) = \pi(f)^*$ holds for all $f \in \mathfrak{A}$ where $\pi(f)^*$ is the adjoint operator of $\pi(f) \in \mathcal{L}(\mathcal{H})$.

A family of representations $(\pi^x, \mathcal{H}_x)_{x \in X}$ is called faithful *whenever the family is injective, i.e., $f = 0$ if and only if $\pi^x(f) = 0$ for all $x \in X$.*

Definition 2.14 (invertible). *Let $\mathfrak{A}$ be a unital Banachalgebra. An element $a \in \mathfrak{A}$ is said to be invertible, if there is an element $b \in \mathfrak{A}$ such that $a \star b = b \star a = e$.*

Remark 2.15. *The inverse b of $a \in \mathfrak{A}$ is unique (if it exists) since $b = b \star (a \star b') = (b \star a) \star b' = b'$.*

Definition 2.16 (spectrum). *Let $\mathfrak{A}$ be a unital Banachalgebra and $a \in \mathfrak{A}$. The set*

$$\sigma(a) := \{\lambda \in \mathbb{C} \,|\, \lambda e - a \textit{ not invertible}\}$$

is called spectrum of a *and* $\rho(a) := \mathbb{C} \setminus \sigma(a)$ *is the* resolvent.

We say that $a \in \mathfrak{A}$ in a $*$-algebra $\mathfrak{A}$ is

- *self-adjoint* if $a^* = a$;
- *normal* if $a^* \star a = a \star a^*$;

2.2 Dynamical systems and the (reduced) C^*-algebra

In this section, we introduce the reduced C^*-algebra associated with a dynamical system. There are different ways to define this C^*-algebra. We follow the construction in the more general setting of groupoid C^*-algebra. The classical reference for groupoid C^*-algebras is [Ren87, Ren91]. The reader is referred also to [Bec16, Section 3.4, Section 3.5] and references therein. The groupoid structure associated with a dynamical system is called transformation group groupoid.
A group G is a set equipped with a composition $\circ : G \times G \to G$, an inverse ${}^{-1} : G \to G$ and a unit $e \in G$ such that

- $\circ$ is associative;
- $g \circ g^{-1} = g^{-1} \circ g = e$ for all $g \in G$.

If, additionally, G is equipped with a topology, G is a topological group if the composition and the inverse are continuous.

Definition 2.17. *The tuple (X, G) is called a (discrete) dynamical system if X is a compact second countable Hausdorff space, G is a countable group (equipped with discrete topology) and there is an $\alpha : G \times X \to X$ is continuous and it satisfies $\alpha(e, x) = x$ and $\alpha\big(g, \alpha(h, x)\big) = \alpha(gh, x)$ for all $x \in X$ and $g, h \in G$.*

Specifically, the group G acts on X by homeomorphism. For simplification, we write gx for $\alpha(g, x)$. Now, we define a C^*-algebra associated with a dynamical system (X, G). A subset $Y \subseteq X$ is called *invariant* if $gY := \{gy | y \in Y\} \subseteq Y$ for all $g \in G$.
Every countable group admits a unique Haar measure λ given by the counting measure $\sum_{g \in G} \delta_g$. A probability measure μ on the Borel σ-algebra of X is called *G-invariant* if $\mu(gF) = \mu(F)$ for all Borel measurable sets $F \subseteq X$ and $g \in G$. Furthermore, μ is said to be *ergodic* if $\mu(Y)$ is zero or one for all invariant subsets $Y \subseteq X$. A dynamical system (X, G) is called *uniquely ergodic* if (X, G) admits exactly one G-invariant measure on X.

Remark 2.18. *A fundamental statement within the theory of dynamical systems is that the space of invariant probability measure (equipped with the weak-$*$ topology) is a convex, closed subset. Its extreme point are given by the ergodic measure. Thus, this set is the closed convex hull of the ergodic measures (Krein-Milman theorem). Note that the existence of invariant measures is an assumption.*

Consider $\mathcal{C}_c(X\times G)$ the set of continuous functions $a: X\times G\to\mathbb{C}$ with compact support. This space is usually equipped with the inductive limit topology (i.e., a net $(a_\iota)_\iota$ converges to $a\in\mathcal{C}_c(X\times G)$ if there is a compact $K\subseteq X\times G$ and a ι_0 such that $\operatorname{supp}(a_\iota)\subseteq K$ for all $\iota\geq\iota_0$ and (a_ι) converges uniformly on K to a).
The set $\mathcal{C}_c(X\times G)$ gets a $*$-algebra if equipped with the convolution

$$(a\star b)(x,g) := \sum_{h\in G} a(x,h)\, b\big(h^{-1}x, h^{-1}g\big) \qquad \textbf{(convolution)}$$
$$a^*(x,g) := \overline{a\big(g^{-1}x, g^{-1}h\big)} \qquad \textbf{(involution)}$$

We can make the $*$-algebra to a C^*-algebra by representing it by suitable operators on the Hilbert space $\ell^2(G)$ and completing the space with the induced norm. For $x\in X$, the left-regular representation $\pi^x:\mathcal{C}_c(X\times G)\to\mathcal{L}\big(\ell^2(G)\big)$ is defined by

$$\big(\pi^x(a)\psi\big)(g) := \sum_{h\in G} a\big(g^{-1}x, g^{-1}h\big)\,\psi(h)\,, \qquad \psi\in\ell^2(G), g\in G,$$

for $a\in\mathcal{C}_c(X\times G)$.

Proposition 2.19. *The family $(\pi^x)_{x\in X}$ defines a faithful $*$-representation. In particular, $\pi^x(a)$ defines a linear, bounded operator on $\ell^2(G)$ for each $a\in\mathcal{C}_c(X\times G)$ and $x\in X$. The operator norm $\|\pi^x(a)\|$ is bounded by*

$$C(a) := \sharp\{g\in G\,|\,\exists x\in X \text{ s.t. } a(x,g)\neq 0\}\cdot\|a\|_\infty .$$

Proof. You can find the proof in the more general case of groupoids in [Ren80, Proposition II.1.1, II.1.4, II.1.9], see also [Bec16, Section 3.4, Section 3.5]. Let us give here a simplified direct proof. That π^x is a $*$-representation follows by direct algebraic computations. Note that all involved sums are finite since $a\in\mathcal{C}_c(X\times G)$ has finite support in G. That $\pi^x(a)$ defines a bounded operator follows by a short computation invoking Cauchy-Schwarz inequality:

$$\begin{aligned}
\|\pi^x(a)\psi\|^2 =& \sum_{\tilde g\in G} \big|(\pi^x(a)\psi)(\tilde g)\big|^2\\
\leq& \sum_{\tilde g\in G}\left(\sum_{h\in G}\big|a\big(\tilde g^{-1}x,\tilde g^{-1}h\big)\big|^{\frac12}\cdot\big|a\big(\tilde g^{-1}x,\tilde g^{-1}h\big)\big|^{\frac12}|\psi(h)|\right)^2\\
\leq& \sum_{\tilde g\in G}\left(\sum_{h\in G}\big|a\big(\tilde g^{-1}x,\tilde g^{-1}h\big)\big|\right)\left(\sum_{h\in G}\big|a\big(\tilde g^{-1}x,\tilde g^{-1}h\big)\big||\psi(h)|^2\right)\\
\leq& C(a)\cdot\left(\sum_{h\in G}\left(\sum_{\tilde g\in G}\big|a\big(\tilde g^{-1}x,\tilde g^{-1}h\big)\big|\right)|\psi(h)|^2\right)\\
\leq& C(a)^2\|\psi\|^2
\end{aligned}$$

Finally it is left to show that the family of representations $(\pi^x)_{x\in X})$ is faithful. Clearly, if $a=0$ then $\pi^x(a)=0$ for all $x\in X$. For the converse direction, let $\pi^x(a)=0$ for all $x\in X$ and assume that $a\neq 0$ which we will show to be a contradiction. By assumption there is a tuple $(y,g)\in X\times G$ such that $a(y,g)\neq 0$. Thus,

$$\big(\pi^y(a)\delta_g\big)(\tilde g) = \sum_{h\in G} a\big(\tilde g^{-1}y,\tilde g^{-1}h\big)\delta_g(h) = a\big(\tilde g^{-1}y,\tilde g^{-1}g\big) \overset{\tilde g=e}{=} a(y,g)\neq 0$$

follows for $\tilde{g} = e$. Hence,

$$\|\pi^y(a)\| \geq \|\pi^y(a)\delta(g)\| = \sqrt{\sum_{\tilde{g}\in G} \big|\big(\pi^y(a)\delta_g\big)(\tilde{g})\big|^2} \geq \big|\big(\pi^y(a)\delta_g\big)(e)\big| = |a(y,g)| > 0$$

is derived being a contradiction. □

Remark 2.20. *It is worth mentioning that $\sharp\{g \in G \,|\, \exists x \in X \text{ s.t. } a(x,g) \neq 0\}$ is finite for $a \in \mathcal{C}_c(X \times G)$ since G is countable and a has compact support.*

With this at hand, we define a norm on $\mathcal{C}_c(X \times G)$ by $\|a\| := \sup_{x\in X} \|\pi^x(a)\|$. This norm is also called *reduced norm*.

Proposition 2.21. *The $*$-algebra $\mathcal{C}_c(X\times G)$ equipped with the norm $\|a\| := \sup_{x\in X} \|\pi^x(a)\|$ is a normed unital $*$-algebra with unit $I := \chi_X \times \delta_e$. Furthermore, the estimate $\|a\|^2 \leq \|a^* \star a\|$ holds.*

Proof. A short computation yields

$$\|a \star b\| = \sup_{x\in X} \|\pi^x(a \star b)\| = \sup_{x\in X} \|\pi^x(a)\, \pi^x(b)\| \leq \sup_{x\in X} \|\pi^x(a)\|\, \|\pi^x(b)\| \leq \|a\|\, \|b\|,$$

implying that $(\mathcal{C}_c(X\times G), \|\cdot\|)$ is a normed algebra. Similarly, the $*$-identity follows as the operator norm fulfills it. For $a \in \mathcal{C}_c(X\times G)$, a short computation leads to $a\star I = I\star a = a$. □

The completion $\mathcal{C}^*_{red}(X\rtimes G)$ of $\mathcal{C}_c(X\times G)$ with respect to the reduced norm is a C^*-algebra.

Remark 2.22. *If G is not discrete or X is not compact, then $\mathcal{C}^*_{red}(X \rtimes G)$ has not a unit. (this is actually a characterization of this C^*-algebra being unital.)*

The previous considerations lead us to the theory of random operator families. Let (X, G) be a discrete dynamical system. Every element $a \in \mathcal{C}^*_{red}(X\rtimes G)$ induces an operator family $A := (A_x)_{x\in X}$ with $A_x := \pi^x(a)$ (convolution operator/integral operator with kernel a).

Proposition 2.23. *Let (X, G) be a discrete dynamical system. Consider an operator family $A := (A_x)_{x\in X}$ with $A_x := \pi^x(a)$ induced by a normal element $a \in \mathcal{C}^*_{red}(X \rtimes G)$. Then the following assertions hold.*

(a) The spectrum $\sigma(a)$ is equal to the union $\overline{\bigcup_{x\in X} \sigma(A_x)}$.

(b) The family of operators is equivariant/covariant, i.e., the equation

$$A_{h.x} \;=\; U_h A_x U_{h^{-1}}$$

holds for all $h \in G$ where $U_h : \ell^2(G) \to \ell^2(G)$, $U_h\psi(g) := \psi(h^{-1}g)$ is unitary.

(c) The map $X \ni x \mapsto A_x$ is strongly continuous on $\mathcal{L}(\ell^2(G))$, i.e., the limit

$$\lim_{y\to x} \big\|\big(A_y - A_x\big)\psi\big\|$$

is equal to zero for all $\psi \in \ell^2(G)$ and $x \in X$.

Proof. (a): For the proof see [NP15]. If π is a $*$-representation of a unital C^*-algebra $\mathfrak{A}$, then $\sigma(\pi(a)) \subseteq \sigma(a)$. (If $a - \lambda \in \mathfrak{A}$ is invertible, then $\pi(a) - \lambda$ is also invertible and so $\rho(a) \subseteq \rho(\pi(a))$.) The converse is proven by contradiction. One shows that $\oplus_{x\in X}\pi^x$: $\mathcal{C}^*_{red}(X \rtimes G) \to \oplus_{x\in X}\mathcal{L}(\ell^2(G))$ is injective, surjective (on the image) and continuous (so a C^*-isomorphism and so it preserves the spectrum. If now $\lambda \in \sigma(a) \setminus \overline{\bigcup_{x\in X} \sigma(A_x)}$, the inverse $\pi^x(a - \lambda)^{-1}$ is well-defined for all $x \in X$ and $\|\pi^x(a - \lambda)^{-1}\|$ is uniformly (in X)

bounded as λ has a positive distance to $\overline{\bigcup_{x\in X}\sigma(A_x)}$. Consequently, $\oplus_{x\in X}\pi^x(a-\lambda)$ is invertible. Since $\oplus_{x\in X}\pi^x$ is an isomorphism, $a-\lambda$ is also invertible, a contradiction.
(b): Let $x \in X$ and $h \in G$. Then, for every $\psi \in \ell^2(G)$ and $g \in G$, a short computation leads to

$$\begin{aligned}
(H_{h.x}\psi)(g) &= \sum_{\tilde{h}\in G} a\left(g^{-1}hx\middle|\, g^{-1}\tilde{h}\right)\cdot\psi(\tilde{h}) \\
&= \sum_{\tilde{h}\in G} a\left((h^{-1}g)^{-1}x\middle|\,(h^{-1}g)^{-1}h^{-1}\tilde{h}\right)\cdot(U_{h^{-1}}\psi)(h^{-1}\tilde{h}) \\
&= (\pi^x(a)(U_{h^{-1}}\psi))(h^{-1}g) \\
&= (U_h A_x U_{h^{-1}}\psi)(g)\,.
\end{aligned}$$

(c): Since $\mathcal{C}_c(X\times G)\subseteq \mathcal{C}^*_{red}(X\rtimes G)$ is a dense subset, it suffices to show the strong continuity for all normal elements of $\mathcal{C}_c(X\times G)$. Let $a\in\mathcal{C}_c(X\times G)$ be normal. First, (i) it is shown that $\big\|\big(\pi^x(a)-\pi^y(a)\big)\psi\big\|$ tends to zero if x for all $\psi\in\mathcal{C}_c(G)$. Secondly, (ii) a 3ε-argument leads to the desired strong continuity of the map $X\ni x\mapsto\pi^x(a)$.
(i): Consider a $\psi\in\mathcal{C}_c(G)$. Then, for $x,y\in X$, the equation

$$\|(\pi^x(a)-\pi^y(a))\,\psi\|^2 = \sum_{g\in G}\left|\sum_{\tilde{h}\in G}\Big(a\big(g^{-1}x\big|g^{-1}\tilde{h}\big)-a\big(g^{-1}y\big|g^{-1}\tilde{h}\big)\Big)\cdot\psi(\tilde{h})\right|^2$$

holds. Since $a\in\mathcal{C}_c(X\times G)$ and $\psi\in\mathcal{C}_c(G)$ are compactly supported, the sums are finite. Thus, the continuity of a implies that the field of bounded normal operators $(\pi^x(a))_x$ is strongly continuous.
(ii): Let $\varphi\in\ell^2(G)$ and $\varepsilon>0$. Since $\mathcal{C}_c(G)\subseteq\ell^2(G)$ is dense, there is a $\psi\in\mathcal{C}_c(G)$ such that $\|\psi-\varphi\|<\frac{\varepsilon}{3C}$ where $C:=\|a\|=\sup_{x\in X}\|\pi^x(a)\|<\infty$. Then choose, by (i), an open neighborhood $U\subseteq X$ of x such that $\big\|\big(\pi^x(a)-\pi^y(a)\big)\psi\big\|<\frac{\varepsilon}{3}$ holds for all $y\in U$. Consequently, the estimate

$$\big\|\big(\pi^x(a)-\pi^y(a)\big)\varphi\big\|\leq\big\|\pi^x(a)\big\|\cdot\big\|\varphi-\psi\big\|+\big\|\big(\pi^x(a)-\pi^y(a)\big)\psi\big\|+\big\|\pi^y(a)\big\|\cdot\big\|\varphi-\psi\big\|<\varepsilon$$

is derived for all $y\in U$. □

Remark 2.24. *If the group G is amenable (or acts amenable on X) then the identity $\sigma(a)=\bigcup_{x\in X}\sigma(A_x)$ holds, [Exe14, NP15].*

Corollary 2.25. *Let $x\in X$ be such that $Orb(x):=\{gx\,|\,g\in G\}$ is dense in X. Then $\sigma(a)=\sigma(A_x)$ holds for every self-adjoint $a\in\mathcal{C}^*_{red}(X\rtimes G)$.*

Proof. Let $y\in X$. Since the orbit $Orb(x)\subseteq X$ is dense, there exists a sequence $(g_n)\subseteq G$ such that $g_nx\to y$. Thus, by the strong continuity we get

$$\sigma(A_y)\subseteq\overline{\lim}_{n\to\infty}\sigma(A_{g_nx}):=\bigcap_{n\in\mathbb{N}}\overline{\big(\bigcup_{m=n}^{\infty}\sigma(A_{g_mx})\big)}.$$

Using the equivariance, we get $\sigma(A_y)\subseteq\sigma(A_x)$. Hence,

$$\sigma(a)=\overline{\bigcup_{y\in X}\sigma(A_y)}\subseteq\sigma(A_x)\subseteq\sigma(A)$$

finishes the proof. □
The section is finished by characterizing the constancy of the spectrum by the minimality of the dynamical system. Recall that a dynamical system (X,G) is minimal if for every $x\in X$, its orbit $Orb(x):=\{gx\,|\,g\in G\}\subseteq X$ is dense.

Proposition 2.26. *Let (X, G) be a discrete dynamical system. Then the following assertions are equivalent.*

(i) The dynamical system (X, G) is minimal.

(ii) For every self-adjoint $a \in \mathcal{C}^*_{red}(X \rtimes G)$, the spectrum $\sigma\big(\pi^x(a)\big)$ is independent of $x \in X$, i.e., $\sigma\big(\pi^x(a)\big) = \sigma\big(\pi^y(a)\big)$ holds for all $x, y \in X$.

(iii) The representation π^x is faithful for every $x \in X$.

Proof. The implication $(i) \Rightarrow (ii)$ is well-known fact invoking Corollary 2.25, see e.g. [CFKS87, BIST89, Jit95, Len99, LS03]. For the proof of the statement, the reader is referred to [Bec16, Theorem 3.6.9] which follows the lines of [LS03, Theorem4.3]. $\square$

2.3 Example: Hamiltonians on $\mathbb{Z}$

2.4 The Pastur-Shubin formula

2.4.1 A short reminder on direct integral theory

2.4.2 IDS for dynamical systems

2.4.3 The classical approach

3 K-theory

4 Gap-labeling theorem

5 Application - The Fibonacci sequence

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