Diffusion determines the recurrent graph

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Motivation

Kac famously posed the question, “Can one hear the shape of a drum?” Mathematically, this question asks whether two domains with unitarily equivalent Laplacians are necessarily congruent. It is well-known that the answer is “No” as long as a sufficiently large class of domains is allowed. A closely related question by Arenstorf replaces the unitary intertwining operator by an order isomorphism and asks if the domains have to be congruent, that is: “Does diffusion determine the body?” [1] Contrary to the original question, the answer to this modified question is “Yes.” We study an analogous problem in a discrete setting, where the domains in Euclidean space and usual Laplace operators are replaced by weighted graphs and graph Laplacians. Similar to the above questions, the objective of our work could be paraphrased as “Does diffusion determine the graph structure?”

Graphs and discrete Dirichlet forms

A weighted graph is a quadruple $(X, b, c, m)$ consisting of a vertex set $X$, a symmetric edge weight $b$, $X \times X \rightarrow [0, \infty)$, a killing term $c: X \rightarrow [0, \infty)$ and measure $m: X \rightarrow (0, \infty)$ such that $b(x, x) = 0$ and $\sum_x b(x, y) < \infty$ for all $x \in X$.

The Dirichlet form with Dirichlet boundary conditions $D$ is defined as the closure of

$$\mathcal{Q}(u, v) = \frac{1}{2} \sum_{x \neq y} b(x, y)(u(x) - u(y))(v(x) - v(y)) + \sum_x c(x)(u(x)v(x) - b(x, x))(v(x) - v(y))$$

on $C_c(X) \subset L^2(X, m)$. Its generator $L$ in $L^2(X, m)$ is called Laplacian with Dirichlet boundary conditions.

Order isomorphisms of $L^p$-spaces

A linear map $U: E \rightarrow F$ between ordered vector spaces is called order isomorphism if it is bijective and $f \geq 0 \iff Uf \geq 0$. Let $(X, m, c, b)$, $i \in \{1, 2\}$, be discrete measure spaces and $p \in [1, \infty]$. If $U: L^p(X, m) \rightarrow L^p(Y, m)$ is an order isomorphism, then there is an associated scaling $h: X \rightarrow (0, \infty)$ and a bijection $\tau: X \rightarrow \tau(X)$ such that

$$Uf = h \cdot (f \circ \tau)$$

for all $f \in L^p(X, m)$.

Metrics on graphs

In a connected graph, the combinatorial graph distance $d(x, y)$ is the minimal number of edges of a path connecting $x$ and $y$. Another useful (pseudometric) asymmetric is given by

$$\rho(x, y) = \inf_{\gamma = (x_0, \ldots, x_n)} \sum_{k=1}^n D(x_k, y_k)$$

where the infimum is taken over all paths $\gamma = (x_0, \ldots, x_n)$ connecting $x$ and $y$.

A generalized ground state transform and recurrent graphs

If $(X, b, 0, m)$ is a graph, $\tau: X \rightarrow Y$ a bijection and $H: X \rightarrow (0, \infty)$ a harmonic function, then the graph $(Y, b_U, m_U)$ defined by $b_U(x, y) = H(x)H(y)b(x, y)$ and $m_U(\tau(x)) = H(x)^2m(x)$ for $x \in X$, $y \in Y$ is called generalized ground state transform of $(X, b, 0, m)$.

If $U$ is an order isomorphism intertwining the Laplacians on $(X, b, 0, m_U)$ and $(X, b_U, 0, m)$, then $(X, b, 0, m)$ is a generalized ground state transform of $(X, b_U, 0, m_U)$ and the harmonic function $H$ in the definition differs from the scaling $h$ only by a constant.

Main theorem [2]

Let $(X, b, 0, m), i \in \{1, 2\}$, be recurrent graphs and let $U: L^2(X, m) \rightarrow L^2(X, m)$ be an order isomorphism intertwining $L_1$ and $L_2$. Then there is a constant $c > 0$ such that

$$b_2(x, y) = cb_1(\tau(x), \tau(y)), \quad m_2(x) = c^2m_1(\tau(x))$$

for all $x, y \in X$.

References