A MINICOURSE ON THE L^p SPECTRUM OF GRAPHS

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Date: Bizerte, March 2014.

1. Introduction

The following notes are compiled for a minicourse held in Bizerte Tunesia in March 2014. The notes are based on joint work with Frank Bauer (Harvard), Bobo Hua (Fudan) and Daniel Lenz (Jena). In particular, the introduction is mainly taken from the with Daniel Lenz [28] on Dirichlet forms on graphs and the survey on intrinsic metrics [27]. The content of the first part of Section 3 is taken from [28] and the second part from [1]. The section on Liouville theorems, Section 5, is extracted from the work with Bobo Hua [20]. Finally, the section on p-independence of the spectrum is taken from the paper with Frank Bauer and Bobo Hua [1].

2. Graphs and Laplacians

In this section we introduce the basic notions for weighted symmetric graphs. We mainly follow the framework developed in [28].

2.1. **Graphs.** Let X be a discrete and countably infinite space and m a measure of full support on X, that is a function $m : X \to (0, \infty)$ which is additively extended to sets via $m(A) = \sum_{x \in A} m(x)$, $A \subseteq X$.

A graph (b, c) over X is a symmetric function $b : X \times X \to [0, \infty)$ with zero diagonal and

$$\sum_{y \in X} b(x, y) < \infty, \quad x \in X,$$

and $c : X \to [0, \infty)$. We say two vertices $x, y \in X$ are *neighbors* if $b(x, y) > 0$. In this case we write $x \sim y$. The function c can describe one way edges to a virtual point at infinity or simply a potential or a killing term. If $c \equiv 0$, then we speak of b as a graph over X. If we already fixed a measure, then we speak of graphs over (X, m).

We say a graph is *connected* if for all $x, y \in X$ there are $x = x_0 \sim \ldots \sim x_n = y$. If a graph is not connected we may restrict our attention to the connected components. Therefore, we assume in the following that the considered graphs are connected.

Given a pair (b, c) an important special case of a measure m is the *normalizing measure*

$$n(x) = \sum_{y \in X} b(x, y) + c(x), \quad x \in X.$$

Another important special case is the *counting measure* $m \equiv 1$.

We say a graph is *locally finite* if every vertex has only finitely many neighbors, that is if the *combinatorial vertex degree* deg is finite at every vertex

$$\deg(x) = \#\{y \in X \mid x \sim y\} < \infty, \qquad \text{for all } x \in X.$$

We speak of a graph with *standard weights* if $b : X \times X \to \{0,1\}$ and $c \equiv 0$. In this case the normalizing measure n equals the combinatorial vertex degree deg. Obviously, by summability of b about vertices, graphs with standard weights are locally finite.

2.2. **Generalized forms and formal Laplacians.** We let $C(X)$ be the set of complex valued functions on X and $C_c(X)$ be the subspace of functions in $C(X)$ of finite support. For a graph (b,c) over X, the generalized quadratic form $\mathcal{Q} : C(X) \to [0,\infty]$ is given by

$$\mathcal{Q}(f) = \frac{1}{2}\sum_{x,y\in X} b(x,y)|f(x)-f(y)|^2 + \sum_{x\in X} c(x)|f(x)|^2$$

with generalized domain

$$\mathcal{D} = \{f \in C(X) \mid \mathcal{Q}(f) < \infty\}.$$

Since $\mathcal{Q}^{\frac{1}{2}}$ is a semi norm and satisfies the parallelogram identity, by polarization $\mathcal{Q}$ yields a semi scalar product on $\mathcal{D}$ via

$$\mathcal{Q}(f,g) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))\overline{(g(x)-g(y))} + \sum_{x\in X} c(x)f(x)\overline{g(x)}.$$

Moreover, for functions in

$$\mathcal{F} = \{f \in C(X) \mid \sum_{y\in X} b(x,y)|f(y)|^2 < \infty \text{ for all } x \in X\},$$

we define the generalized Laplacian $\mathcal{L} : \mathcal{F} \to C(X)$ by

$$\mathcal{L}f(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(f(x)-f(y)) + \frac{c(x)}{m(x)}f(x).$$

In [15, Lemma 4.7], a *Green's formula* is shown for functions $f \in \mathcal{F}$ and $\varphi \in C_c(X)$

$$\frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))\overline{(g(x)-g(y))} + \sum_{x\in X} c(x)f(x)\overline{g(x)}$$

$$= \sum_{x\in X} \mathcal{L}f(x)\overline{\varphi(x)} = \sum_{x\in X} f(x)\overline{\mathcal{L}\varphi(x)}.$$

Denote

$$df(x,y) = f(x) - f(y), \qquad f \in C(X).$$

We call $f \in \mathcal{F}$ a *solution* (respectively *subsolution* or *supersolution*) for $\lambda \in \mathbb{R}$ if $(\mathcal{L}-\lambda)f = 0$ (respectively $(\mathcal{L}-\lambda)f \leq 0$ or $(\mathcal{L}-\lambda)f \geq 0$). A solution (respectively subsolution or supersolution) for $\lambda = 0$ is called a harmonic (respectively subharmonic or superharmonic).

We say a function $f \in C(X)$ is *positive* if $f \geq 0$ and non-trivial and *strictly positive* if $f > 0$.

A *Riesz space* is a linear space equipped with a partial ordering which is consistent with addition, scalar multiplication and where the maximum and the minimum of two functions exist.

An important fact that is needed in the subsequent is that in order to study existence of non-constant (respectively non-zero) solutions for $\lambda \leq 0$ in a Riesz space, it suffices to study positive subharmonic functions.

Lemma 2.1. *Let (b, c) be a connected graph over (X, m) and $\mathcal{F}_0 \subseteq \mathcal{F}$ be a Riesz space. If there are no non-constant positive subharmonic functions in $\mathcal{F}_0$, then there are no non-constant solutions for $\lambda \leq 0$ in $\mathcal{F}_0$. In particular, any constant solution to $\lambda < 0$ is zero.*

Proof. For a solution f to $\lambda \leq 0$ the positive part $f_+ = f \vee 0$, the negative part $f_- = -f \vee 0$ and the modulus $|f| = f_+ + f_-$ can be seen to be non-negative subharmonic functions. Thus, the statement follows from connectivity. The 'in particular' is obvious. □

2.3. **Dirichlet forms and their generators.** The form and Laplacian introduced above are defined on spaces with very few properties. Next, we will consider these objects restricted to suitable Hilbert and Banach spaces.

Let $\ell^p(X, m)$ be the canonical complex-valued ℓ^p-spaces, $p \in [1, \infty]$, with norms

$$\|f\|_p = \Big(\sum_{x \in X} |f(x)|^p m(x)\Big)^{\frac{1}{p}}, \; p \in [1, \infty),$$

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

As $\ell^\infty(X, m)$ does not depend on m we also write $\ell^\infty(X)$. For $p = 2$, we have a Hilbert space $\ell^2(X, m)$ with scalar product

$$\langle f, g \rangle = \sum_{x \in X} f(x)\overline{g(x)} m(x), \qquad f, g \in \ell^2(X, m),$$

and we denote the norm $\|\cdot\| = \|\cdot\|_2$.

2.3.1. *Dirichlet forms.* Restricting the form $\mathcal{Q}$ to $\mathcal{D} \cap \ell^2(X, m)$, we see by Fatou's lemma that this restriction is lower semi-continuous and, thus, closed. Hence, the restriction of $\mathcal{Q}$ to $C_c(X)$ is closable.

The $Q = Q_{b,c}$ be the quadratic form given by

$$D(Q) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}, \quad \text{where } \|\cdot\|_{\mathcal{Q}} = \big(\mathcal{Q}(\cdot) + \|\cdot\|^2\big)^{\frac{1}{2}}$$

$$Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)|\varphi(x) - \varphi(y)|^2 + \sum_{x \in X} c(x)|\varphi(x)|^2, \qquad f \in D(Q).$$

It can be seen that Q is a *Dirichlet form* (see [8, Theorem 3.1.1] for general theory and for a proof in the graph setting see [39, Proposition 2.10]), that is for any $f \in D(Q)$ we have $0 \vee f \wedge 1 \in D(Q)$ and

$$Q(0 \vee f \wedge 1) \leq Q(f).$$

Obviously, Q is *regular*, that is $C_c(X) \cap D(Q)$ is dense in $D(Q)$ with respect to $\| \cdot \|_Q$ and dense in $C_c(X)$ with respect to $\| \cdot \|_\infty$.

As it turns out, by [28, Theorem 7], all regular Dirichlet forms are given in this way. A fact which can be also derived directly from the Beurling-Deny representation formula [8, Theorem 3.2.1 and Theorem 5.2.1].

Theorem 2.2 (Theorem 7 in [28])**.** *If q is a regular Dirichlet on $\ell^2(X,m)$, then there is a graph (b,c) such that $q = Q_{b,c}$.*

2.3.2. *Markovian semigroups and their generators.* By general theory (see e.g. [44, Satz 4.14]), Q yields a positive selfadjoint operator L with domain $D(L)$ viz

$$Q(f,g) = \langle L^{\frac{1}{2}} f, L^{\frac{1}{2}} g \rangle, \qquad f,g \in D(Q).$$

By the second Beurling-Deny criterion L gives rise to a Markovian semigroup e^{-tL}, $t > 0$, which extends consistently to all $\ell^p(X,m)$, $p \in [1,\infty]$, and is strongly continuous for $p \in [1,\infty)$. *Markovian* means that for functions $0 \leq f \leq 1$, one has $0 \leq e^{-tL} f \leq 1$.

We denote the generators of e^{-tL} on $\ell^p(X,m)$, $p \in [1,\infty)$, by L_p, that is

$$D(L_p) = \left\{ f \in \ell^p(X,m) \mid g = \lim_{t \to 0} \frac{1}{t}(I - e^{-tL}) f \text{ exists in } \ell^p(X,m) \right\}$$

$$L_p f = g$$

and L_∞ is defined as the dual of L_1. Simultaneously, the resolvents $G_z = (L_2 - z)^{-1}$, $z \in \{ w \in \mathbb{C} \mid \Re w < 0 \}$ (the open left half plane) extend consistently to $\ell^p(X,m)$, $p \in [1,\infty]$. By the spectral theorem the Laplace transform for the resolvent

$$G_z f = \int_0^\infty e^{zt} T_t f dt,$$

holds for f in ℓ^2. By density and duality arguments, this formula extends to f in ℓ^p, $p \in [1,\infty)$, in the strong sense and to $p = \infty$ in the weak sense. This shows that the resolvents $(L_p - z)^{-1}$ are consistent on ℓ^p for $z \in \{ w \in \mathbb{C} \mid \Re w < 0 \}$.

Theorem 2.3 (Theorem 9 in [28])**.** *Let (b,c) be a graph over (X,m). Let $p \in [1,\infty]$ be given. Then,*

$$L_p f = \mathcal{L} f \text{ for any } f \in D(L_p).$$

Proof. Let $f \in D(L_p)$ be given. Then, $g := (L_p + \alpha)f$ exists and belongs to $\ell^p(X,m)$. By the lemma below, $f = (L_p + \alpha)^{-1}g$ below solves

$$(\mathcal{L} + \alpha)f = g = (L_p + \alpha)f$$

and we infer the statement. □

Lemma 2.4. *Let (b,c) be a graph over (X,m). Let $p \in [1,\infty]$ be given. For any $g \in \ell^p(X,m)$, the function $u := (L_p + \alpha)^{-1}g$ belongs to the set $\mathcal{F}$ on which $\mathcal{L}$ is defined and solves $(\mathcal{L} + \alpha)u = g$.*

Proof. We first consider the case $p = 2$. If suffices to consider the case $f \geq 0$. Choose an increasing sequence (K_n) of finite subsets of X with $\bigcup K_n = X$ and let g_n be the restriction of g to K_n. Then, (g_n) converges monotonously increasing to g in $\ell^2(X,m)$ and consequently $(L + \alpha)^{-1}g_n$ converges monotonously increasing to u. Thus, by monotone convergence of solutions, we can assume without loss of generality that g has compact support contained in K_1. By convergence of resolvents, $u_n := (L_{K_n}^{(D)} + \alpha)^{-1}g$ then converges increasingly to $u := (L + \alpha)^{-1}g$. Moreover, u_n satisfies $(\mathcal{L} + \alpha)u_n = g$ on K_n. Thus, the statement follows by monotone convergence of solutions.

We now turn to general $p \in [1,\infty]$. Again, it suffices to consider the case $g \geq 0$. Choose an increasing sequence (K_n) of finite subsets of X with $\bigcup K_n = X$ and let g_n be the restriction of g to K_n. Then, $u_n := (L_p + \alpha)^{-1}g_n$ converges to u. Moreover, as g_n belongs to $\ell^2(X,m)$ consistency of the resolvents gives $u_n = (L + \alpha)^{-1}g_n$. Now, on the $\ell^2(X,m)$ level we can apply the considerations for $p = 2$ to obtain

$$(\mathcal{L} + \alpha)u_n = (\mathcal{L} + \alpha)(L + \alpha)^{-1}g_n = g_n.$$

Taking monotone limits now yields the statement. □

2.3.3. *Graphs with standard weights.* Two important special cases are graphs with standard weights, that is $b : X \times X \to \{0,1\}$ and $c \equiv 0$.

For the counting measure $m \equiv 1$, we denote the operator L on $\ell^2(X) = \ell^2(X,1)$ by Δ which operates as

$$\Delta f(x) = \sum_{y \sim x}(f(x) - f(y)), \quad f \in D(\Delta),\, x \in X.$$

We will see below in Section 2.3.4 that Δ is bounded if and only if deg is bounded.

For the normalized measure $n = \deg$, we call the operator L on $\ell^2(X,\deg)$ the *normalized Laplacian* and denoted it by Δ_n. The operator Δ_n acts as

$$\Delta_n f(x) = \frac{1}{\deg(x)}\sum_{y \sim x}(f(x) - f(y)), \quad f \in \ell^2(X,\deg),\, x \in X,$$

and, as it can also be seen below, Δ_n is always bounded by 2.

2.3.4. *Boundedness of the operators.* We next comment on the boundedness of the form Q and the operator L. The theorem below is taken from [16] and an earlier version can be found in [29, Theorem 11].

Theorem 2.5 (Theorem 9.3 in [16]). *Let (b, c) be a graph over (X, m). Then the following are equivalent:*

(i) $X \to [0, \infty)$, $x \mapsto \frac{1}{m(x)}\Big(\sum_{y\in X} b(x, y) + c(x)\Big)$ *is a bounded function.*
(ii) L_p *is bounded for some* $p \in [1, \infty]$.
(iii) L_p *is bounded for all* $p \in [1, \infty]$.

Specifically, if the function in (i) is bounded by $D < \infty$, then $Q \leq 2D$ and $\|L\|_p \leq 2D$, $p \in [1, \infty]$.

Proof. Note that we have seen that L_p is a restriction of $\mathcal{L}$ by what we have discussed above. Thus, it suffices to consider boundedness of $\mathcal{L}$.

Assume that (i) is satisfied. Then, we see that $\mathcal{L}$ is a bounded operator on ℓ^∞. Since $\mathcal{L}$ is symmetric, we get, by duality, that $\mathcal{L}$ is bounded on ℓ^1. Applying the Riesz-Thorin theorem, we get (iii), resp. (ii). Assume, conversely, that (ii) is fulfilled. Again, by duality and symmetry, $\mathcal{L}$ is also a bounded operator on ℓ^q, where $1 = \frac{1}{p} + \frac{1}{q}$. Using interpolation once more, we get that $\mathcal{L}$ is bounded on ℓ^2. Hence, for each $x \in V$, we have the existence of $C \geq 0$ such that

$$\langle \mathcal{L}\delta_x, \delta_x\rangle \leq Cm(x),$$

which gives (i). □

2.3.5. *The compactly supported functions as a core.* It shall be observed that $C_c(X)$ is in general not included in $D(L)$. Indeed, one can give a characterization for this situation. The proof is rather immediate and we refer to [28, Proposition 3.3] or [14, Lemma 2.7.] for a reference.

Lemma 2.6. *Let (b, c) be a graph over (X, m). Then the following are equivalent:*

(i) $C_c(X) \subseteq D(L)$.
(ii) $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$
(iii) *The functions* $X \to [0, \infty)$, $y \mapsto \frac{1}{m(y)} b(x, y)$ *are in* $\ell^2(X, m)$.

In particular, the above assumptions are satisfied if the graph is locally finite or

$$\inf_{y\sim x} m(y) > 0, \qquad x \in X.$$

Moreover, either of the above assumptions implies $\ell^2(X, m) \subseteq \mathcal{F}$.

Proof. The equivalence of (ii) and (iii) follows from the abstract definition of the domain of L as $D(L) = \{f \in D(Q) \mid \text{there is } l \in \ell^2(X, m) \text{ such that } \langle l, \varphi\rangle = Q(f, \varphi) \text{ for all } \varphi \in D(Q)\}$. The equivalence of (i) and (ii) is a direct calculation, see [28, Proposition 3.3] and the 'in particular' statements are also immediate, see [28, 14]. □

3. Uniformly positive measures

In this section we discuss two results for graphs which seem to have no analogue in the non-discrete setting. The condition below is about (X, m) only as a measure space. We say the measure m is *uniformly positive* if

(A) $\inf_{x\in X} m(x) > 0$.

For example this holds if m is constant as in the case of the counting measure or deg.

3.1. **Liouville type theorem.**

This condition yields a result for existence of certain solutions in $\ell^p(X, m)$ which in contrast to the metric result above includes the case $p = 1$.

Theorem 3.1 (Lemma 3.2 in [28])**.** *Let (b, c) be a graph over (X, m) with uniformly positive measure. Then any positive subharmonic function in $\ell^p(X, m)$, $p \in [1, \infty)$, is zero.*

Proof. Let $f \in \ell^p(X, m)$, for some $p \in [1, \infty)$, be positive and subharmonic. Also, let $x \in X$ such that $f(x) > 0$. Then, $\mathcal{L}f(x) \le 0$ gives using $f \ge 0$

$$f(x) \le \frac{1}{\sum_{y\in X} b(x, y)} \sum_{y\in X} b(x, y)f(y)$$

Thus, there must be $y \sim x$ such that $f(x) \le f(y)$. Proceeding inductively there is a sequence (x_n) of vertices such that $0 < f(x) \le f(x_n) \le f(x_{n+1})$, $n \ge 0$. By (A) together with $f \in \ell^p(X, m)$, this implies $f \equiv 0$. □

Remark. Note that the theorem also holds under the weaker assumption

(A*) $\sum_{n=1}^{\infty} m(x_n) = \infty$ for all infinite paths (x_n).

Theorem 3.2 (Theorem 5 in [28])**.** *Let (b, c) be a graph over (X, m) such that (A) holds. Then, for any $p \in [1, \infty)$ the operator L_p is the restriction of $\mathcal{L}$ to*

$$D(L_p) = \{u \in \ell^p(X, m) : \mathcal{L}u \in \ell^p(X, m)\}.$$

Proof. Let f be positive and subharmonic. Then, $\mathcal{L}f(x) \le 0$ evaluated at some x gives, using $f \ge 0$,

$$f(x) \le \frac{1}{\sum_{y\in X} b(x, y)} \sum_{y\in X} b(x, y)f(y)$$

Thus, whenever there is $x' \sim x$ with $f(x') < f(x)$ there must be $y \sim x$ such that $f(x) < f(y)$. Such x' and x exist whenever f is non-constant. Letting $x_0 = x$, $x_1 = y$ and proceeding inductively there is a sequence (x_n) of vertices such that $0 < f(x) < f(x_n) < f(x_{n+1})$, $n \ge 0$. Now, (A) implies that f is not in $\ell^p(X, m)$. On the other hand, if f is constant, then (A) again implies $f \equiv 0$. □

Theorem 3.3 (Theorem 6 in [28]). *Let (b,c) be a graph over (X,m). Assume*

$$\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$$

and (A). Then the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially selfadjoint and the domain of L is given by

$$D(L) = \{u \in \ell^2(X,m) : \mathcal{L}u \in \ell^2(X,m)\}.$$

Proof. As $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, we can define the minimal operator $L_{\min}$ to be the restriction of $\mathcal{L}$ to

$$D(L_{\min}) := C_c(X)$$

and the maximal operator $L_{\max}$ to be the restriction of $\mathcal{L}$ to

$$D(L_{\max}) := \{u \in \ell^2(X,m) : \mathcal{L}u \in \ell^2(X,m)\}.$$

The previous proposition gives

$$\langle u, L_{\min} v\rangle = Q(u,v)$$

for all $u,v \in C_c(X)$ which extends to all $u \in D(Q)$ and $v \in C_c(X)$. Thus, $L_{\min}$ is a restriction of L in this case.
Moreover, by Green's formula above

$$\langle u, L_{\min} v\rangle = \sum_{x\in X} \mathcal{L}u(x)v(x)m(x)$$

for all $v \in C_c(X)$ and $u \in \ell^2(X,m)$. Thus,

$$L_{\min}^* = L_{\max}.$$

Thus, essential selfadjointness of $L_{\min}$ is equivalent to selfadjointness of $L_{\max}$. This in turn is equivalent to $L = L_{\max}$ (as we have $L \subseteq L_{\max}$ by Theorem 3.2). As (A) and Theorem 3.2 yield $D(L) = \{u \in \ell^2(X,m) : \mathcal{L}u \in \ell^2(X,m)\}$, we infer $L = L_{\max}$ and essential selfadjointness of the restriction of L to $C_c(X)$ $(= L_{\min})$ follows. □

3.2. **Spectral inclusion.** In this section we prove the spectral inclusion $\sigma(L_2) \subseteq \sigma(L_p)$ under the assumption of uniformly positive measure. We notice that $\inf_{x\in X} m(x) > 0$ implies

$$\ell^p \subseteq \ell^q, \quad 1 \le p \le q \le \infty.$$

Moreover, by the theorem above we know the domains of the generators L_p in this case explicitly, namely

$$D(L_p) = \{f \in \ell^p \mid \mathcal{L}f \in \ell^p\}.$$

In particular, this gives

$$D(L_p) \subseteq D(L_q), \quad 1 \le p \le q \le \infty.$$

Furthermore, it can be checked directly that $C_c(X) \subseteq D(L_p)$.

The following theorem shows that under an assumption on the measure one spectral inclusion holds without any volume growth assumptions.

Theorem 3.4. *Let (b,c) be a graph over (X,m) such that (A) holds, then, for any $p \in [1,\infty]$,*

$$\sigma(L_2) \subseteq \sigma(L_p).$$

Before we come to the proof of the theorem we have to establish consistency of resolvents on the intersection of the resolvent sets.

Lemma 3.5. *Let (b,c) be a graph over (X,m) such that (A) holds. Then, for all $1 \le p \le q \le \infty$ and all $z \in \rho(L_p) \cap \rho(L_q)$ the resolvents $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are consistent on $\ell^p = \ell^p \cap \ell^q$.*

Proof. Let $1 \le p < q \le \infty$ and $z \in \rho(L_p) \cap \rho(L_q)$. As $D(L_p) \subseteq D(L_q)$ and $L_p = L_q$ on $D(L_p)$, we have for all $f \in \ell^p \subseteq \ell^q$

$$(L_q - z)(L_p - z)^{-1} f = (L_p - z)(L_p - z)^{-1} f = f$$

Hence, $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are consistent on $\ell^p = \ell^p \cap \ell^q$. □

Proof of Theorem 3.4. Let $p \in [1,2]$. By the lemma above the resolvents $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are consistent for $z \in \rho(L_p) = \rho(L_q)$ on ℓ^p. By the Riesz-Thorin interpolation theorem $(L_q - z)^{-1}$ is bounded on ℓ^2. We will show that $(L_q - z)^{-1}$ is an inverse of $(L_2 - z)$ for $z \in \rho(L_p) = \rho(L_q)$. So, let $z \in \rho(L_q)$. As $D(L_2) \subseteq D(L_q)$, $\ell^2 \subseteq \ell^{p^*}$ and L_2, L_q are restrictions of $\mathcal{L}$ we have for $f \in D(L_2)$

$$(L_q - z)^{-1}(L_2 - z) f = (L_q - z)^{-1}(L_q - z) f = f.$$

Secondly, let $f \in \ell^2$ and (f_n) be such that $f_n \in \ell^p$ and $f_n \to f$ in ℓ^2. As $(L_q - z)^{-1}$ is ℓ^2-bounded, $(L_q - z)^{-1} f_n \to (L_q - z)^{-1} f$, $n \to \infty$, in ℓ^2. By the lemma above $(L_q - z)^{-1} f_n = (L_p - z)^{-1} f_n \in D(L_p) \subseteq D(L_2)$, and, therefore,

$$(L_2 - z)(L_q - z)^{-1} f_n = (L_q - z)(L_q - z)^{-1} f_n = f_n \to f, \quad n \to \infty,$$

in ℓ^2. Since, L_2 is closed we infer $(L_q - z)^{-1} f \in D(L_2)$ and $(L_2 - z)(L_q - z)^{-1} f = f$. Hence, $(L_q - z)^{-1}$ is an inverse of $(L_2 - z)$ and, thus, $z \in \rho(L_2)$. □

4. Intrinsic metrics

In this subsection we discuss the notion of intrinsic metrics for a weighted graph. Such metrics have proven to be very effective in the context of strongly local Dirichlet forms, see e.g. [43]. Recently, this concept was generalized to all regular Dirichlet forms. It was first systematically studied by Frank/Lenz/Wingert in [7], (see also [34] for an earlier mention of the criterion for certain non-local forms).

4.1. **Definition of intrinsic metrics.** We call a symmetric map $\rho : X \times X \to [0,\infty)$ with zero diagonal a *pseudo metric* if it satisfies the triangle inequality. By [7, Lemma 4.7, Theorem 7.3] it can be seen that the following definition coincides with the definition of an intrinsic metric for general regular Dirichlet forms, [7, Definition 4.1].

Definition 4.1. A pseudo metric ρ is called an *intrinsic metric* with respect to a graph (b,c) over (X,m) if for all $x \in X$

$$\sum_{y\in X} b(x,y)\rho^2(x,y) \leq m(x).$$

Similar definitions of such metrics were also introduced in the context of graphs or jump processes under the name adapted metrics in [5, 6, 13, 21, 23, 34].

4.2. **Examples and relations to other metrics.**

4.2.1. *The degree path metric.* A first specific example of an intrinsic metric was introduced by Huang, [21, Definition 1.6.4] and it also appeared in [4]. Consider the pseudo metric $\rho_0 : X \times X \to [0,\infty)$ given by

$$\rho_0(x,y) = \inf_{x=x_0\sim x_1\sim\ldots\sim x_n=y} \sum_{i=1}^{n} \Big(\mathrm{Deg}(x_{i-1}) \vee \mathrm{Deg}(x_i)\Big)^{-\frac{1}{2}}, \qquad x \in X,$$

where $\mathrm{Deg} : X \mapsto (0,\infty)$ is the *weighted vertex degree* given by

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y\in X} b(x,y), \qquad x \in X.$$

We call such a metric that minimizes sums of weights over paths of edges a *path metric.*

It can be seen directly that ρ_0 defines an intrinsic metric for the graph (b,c) over (X,m)

$$\sum_{y\in X} b(x,y)\rho_0^2(x,y) \leq \sum_{y\in X} \frac{b(x,y)}{\mathrm{Deg}(x)\vee \mathrm{Deg}(y)} \leq \sum_{y\in X} \frac{b(x,y)}{\mathrm{Deg}(x)} = m(x).$$

There is the following intuition behind the definition of ρ_0. Consider the Markov process $(X_t)_{t\geq 0}$ associated to the semigroup e^{-tL} via

$$e^{-tL}f(x) = \mathbb{E}_x(f(X_t)),$$

where $\mathbb{E}_x$ is the expected value conditioned on the process starting at x. The random walker modeled by this process jumps from a vertex x to a neighbor y with probability $b(x,y)/\sum_z b(x,z)$. Moreover, the probability of not having left x at time t is given by

$$\mathbb{P}_x(X_s = x, 0 \leq s \leq t) = e^{-\mathrm{Deg}(x)t}.$$

Qualitatively, this indicates that the larger $\mathrm{Deg}(x)$, the faster the random walker leaves x. Looking at the definition of $\rho_0(x,y)$, the larger the degree of either x or y the closer are the two vertices. Combining these two observations, we see that the faster the random walker jumps along an edge the shorter the edge is with respect to ρ_0. Of course, the jumping time along an edge connecting x to y is not symmetric and depends on whether one jumps from x to y or from y to x as the degrees of x and y can be very different. In order to get a symmetric function, ρ_0 favors the vertex with the larger degree and the faster jumping time.

There is a direct analogy to the Riemannian setting in terms of mean exit times of small balls. Consider a small ball B_r of radius r on a d-dimensional Riemannian manifold, the first order term of the mean exit time of B_r is $r^2/2d$, [37]. Now, on a locally finite graph for a vertex x a 'small' ball with respect to ρ_0 can be thought to have radius $r = \inf_{y\sim x} \rho(x,y)/2$, namely this ball contains only the vertex itself. Now, computing the mean exit time of this ball gives $1/\mathrm{Deg}(x) \geq r^2$, where equality holds whenever $\mathrm{Deg}(x) = \max_{y\sim x} \mathrm{Deg}(y)$.

4.2.2. *The combinatorial graph distance.* We call the path metric defined by

$$d(x,y) = \min \#\{n \in \mathbb{N} \mid \text{there are } x_0,\ldots,x_n \text{ with } x = x_0 \sim \ldots \sim x_n = y\}$$

the *combinatorial graph distance*. If one considers now a graph (b,c) over (X,m) with $m = n$ being the normalizing measure we find that

$$\sum_{y\in X} b(x,y)d(x,y)^2 = \sum_{y\in X} b(x,y) \leq n(x).$$

Hence, d is always an intrinsic metric for (b,c) over (X,n). In particular, for $c \equiv 0$,

$$\mathrm{Deg}(x) = \frac{1}{n(x)} \sum_{y\in X} b(x,y) = 1$$

for all $x \in X$ which readily gives

$$d = \rho_0$$

in the case of the normalizing measure n. Specifically, this is true for the normalized Laplacian Δ_n.

On the other hand, for a graph with standard weights and the counting measure associated to the Laplacian Δ, the combinatorial graph distance d satisfies

$$\sum_{y\sim x} b(x,y)d(x,y)^2 = \deg(x)$$

for all $x \in X$. Hence, there is an intrinsic metric equivalent to the combinatorial graph distance if and only if the combinatorial vertex degree deg is bounded, and, in which case Δ is a bounded operator. Moreover, in this case

$$\mathrm{Deg} = \deg$$

and, therefore, by connectedness

$$\rho_0 \le d \le D^{\frac{1}{2}} \rho_0,$$

with $D = \sup_{x \in X} \deg(x)$.

4.2.3. *Comparison to the strongly local case.* An important difference to the case of strongly local Dirichlet forms is that in the graph case there is no maximal intrinsic metric. For example for a Riemannian manifold M the Riemannian distance d_M is the maximal C^1 metric ρ_M that satisfies

$$|\nabla \rho_M(o, \cdot)| \le 1,$$

for all $o \in M$,where ∇_M is the Riemannian gradient. In fact, d_M can be recovered by the formula

$$d_M(x, y) = \sup\{f(x) - f(y) \mid f \in C_c^\infty(M)\ |\nabla_M f| \le 1\}, \quad x, y \in X.$$

Now, for discrete spaces the maximum of two intrinsic metrics is not necessarily an intrinsic metric. In particular, consider the pseudometric σ

$$\sigma(x, y) = \sup\{f(x) - f(y) \mid f \in \mathcal{A}\}, \qquad x, y \in X,$$

where

$$\mathcal{A} = \big\{f : X \to \mathbb{R} \mid \sum_{y \in X} b(x, y)|f(x) - f(y)|^2 \le m(x) \text{ for all } x \in X\big\}$$

As discussed for the Riemannian case above, the strongly local analogue d_M of σ defines the maximal intrinsic metric in the strongly local case, but σ is in general not an intrinsic metric in the graph case.

A basic example where this can be seen immediately can be found in [7, Example 6.2]. More generally, this can be seen for arbitrary tree graphs associated to the operator Δ. In this case $\sigma = \frac{1}{2} d$ and by discussion above we already know that the combinatorial graph distance d is typically not an intrinsic metric for Δ.

4.2.4. *Resistance metrics.* Another important metric appears in the context of resistance metrics. Let $r : X \times X \to [0, \infty)$ be given by

$$r(x, y) = \sup\{f(x) - f(y) \mid f \in \mathcal{D},\ \mathcal{Q}(f) \le 1\}, \qquad x, y \in X.$$

Indeed, r is the square root of the resistance metric as it appears e.g. in [32], see [9, Theorem 3.20]. In [9] this metric is related to intrinsic metrics.

Theorem 4.2 (Theorem 3.14 in [9]). *Let b be a connected graph over X. Then*

$$r = \sup\{\rho \mid \text{intrinsic metric for } b \text{ over } (X,m) \text{ with } m(X) = 1\}.$$

4.2.5. *Another path metric.* Colin de Verdiere/Torki-Hamza/Truc [2] studied a path pseudo metric δ which is given as

$$\delta(x,y) = \inf_{x=x_0\sim\ldots\sim x_n=y} \sum_{i=0}^{n-1} \Big(\frac{m(x)\wedge m(y)}{b(x,y)}\Big)^{\frac{1}{2}}, \qquad x,y\in X.$$

This metric is equivalent to the intrinsic metric ρ_0 if and only if the combinatorial vertex degree deg is bounded on the graph.

4.3. **A Hopf-Rinow theorem.** We shall stress that in general an intrinsic metric ρ (and in particular ρ_0) is not a metric and (X,ρ) might not even be locally compact, as can be seen from examples in [24, Example A.5]. However, for locally finite graphs and path metrics such as ρ_0 the situation is much tamer. For example one can show a Hopf-Rinow type theorem. Note that we do not need that the metrics are intrinsic for this section.

Theorem 4.3 (Theorem A1 in [16]). *Let (b,c) be a connected graph over (X,m) and ρ be a path metric. Then, the following are equivalent:*

(i) *(X,ρ) is complete as a metric space.*
(ii) *(X,ρ) is geodesically complete, that is any infinite path (x_n) of vertex that realizes the distance has infinite length.*
(iii) *Every distance ball is finite.*
(iv) *Every bounded and closed set is compact.*

Remark. (a) Anytime a path pseudo metric d induces the discrete topology on X the following implications hold: (iii)⇔(iv)⇒(i)⇒(ii). This is the case if and only if $\inf_{y\sim x}\sigma(x,y) > 0$ for all $x\in X$. In fact, (iv)⇒(i) holds for general metric spaces. The stronger assumption of local finiteness is needed for the implications (ii)⇒(i), (i)⇒(iii) (or (iv)) and (ii)⇒(iii) (or (iv)).

(b) A similar statement as (i)⇒(iii) can also be found in [36].

We prove the theorem in several steps through the following lemmas.

Lemma 4.4. *Let (b,c) be a connected locally finite graph over (X,m) and ρ be a path metric. Then, the following hold:*

(a) *(X,ρ) is a discrete metric space. In particular, (X,ρ) is locally compact.*
(b) *A set is compact in (X,ρ) if and only if it is finite.*

Proof. Local finiteness implies that for all $x\in X$ there is an $r>0$ such that $\rho(x,y)>r$ for all $y\in X$ with $y\sim x$. First, by the definition of ρ, we have that for all $x,z\in X$ there is $y\sim x$ with $\rho(x,y)\le\rho(x,z)$. Thus, $\rho(x,y)=0$ implies $x=y$, therefore, d is a metric. Second, it yields that $B_r(x) = \{x\}$ and $\{x\}$ is an open set which shows (a). From this we conclude that for any infinite set U the cover $\{\{x\} \mid x \in U\}$ has no finite subcover. The other direction of (b) is clear. $\square$

Lemma 4.5. *Let (b, c) be a connected locally finite graph over (X, m) and ρ be a path metric. Then, there exists an infinite geodesic of bounded length.*

Proof. Let $o \in X$ be the center of the infinite ball B_r of radius r and let d be the combinatorial graph distance. Let P_n, $n \geq 0$, be the set of finite paths $(x_0, \ldots, x_k)$ such that $x_0 = o$, $x_i \neq x_j$ for $i \neq j$, $\rho(x_k, o) = n$ and $d(x_j, o) \leq n$ for $j = 0, \ldots, k$. Let

$$l((x_0, \ldots, x_k)) = \sum_{i=0}^{k-1} \rho(x_i, x_{i+1}).$$

Claim: $\Gamma_n = \{\gamma \in P_n \mid \gamma \text{ geodesic with respect to } \rho,\ l(\gamma) \leq r\} \neq \emptyset$ for all $n \geq 0$.

Proof of the claim: The set P_n is finite by local finiteness of the graph and, thus, contains a minimal element $\gamma = (x_0, \ldots, x_K)$ with respect to the length l, i.e., for all $\gamma' \in P_n$ we have $l(\gamma') \geq l(\gamma)$. Then, γ is a geodesic: for every path $(x_0', \ldots, x_M')$ with $x_0' = o$ and $x_M' = x_K$, we let $m \in \{n, \ldots, M\}$ be such that $(x_0', \ldots, x_m') \in P_n$. By the minimality of γ we infer

$$l((x_0', \ldots, x_M')) \geq l((x_0', \ldots, x_m')) \geq l(\gamma).$$

It follows that γ is a geodesic. Clearly, $l(\gamma) \leq r$, as otherwise $B_r \subseteq \{y \in X \mid d(y, o) \leq n - 1\}$ which would imply the finiteness of B_r by the local finiteness of the path space. Thus, $\gamma \in \Gamma_n$ which proves the claim.

We inductively construct an infinite geodesic (x_k) with bounded length: We set $x_0 = o$. Since $\Gamma_n \neq \emptyset$, there is a geodesic in Γ_n for every $n \geq 0$ such that x_0 is a subgeodesic. Suppose we have constructed a geodesic $(x_1, \ldots, x_m)$ such that for all $n \geq m$ there is a geodesic in Γ_n that has $(x_1, \ldots, x_m)$ as a subgeodesic. By local finiteness x_m has finitely many neighbors. Thus, there must be a neighbor x_{m+1} of x_m such that for infinitely many n the path $(x_0, \ldots, x_m, x_{m+1})$ is a subpath of a geodesic in Γ_n. However, a subpath of geodesic is a geodesic. Thus, there is an infinite geodesic $\gamma = (x_k)_{k \geq 0}$ with $l(\gamma) = \lim_{n \to \infty} l((x_0, \ldots, x_n)) \leq r$ as $(x_0, \ldots, x_n) \in \Gamma_n$ for all $n \geq 0$. $\square$

Proof of Theorem 4.3. The fact that (X, ρ) is a discrete metric space follows from Lemma 4.4. We now turn to the proof of the equivalences. We start with (i)$\Rightarrow$(ii). If there is a bounded geodesic, then it is a Cauchy sequence. Since a geodesic is a path it is not eventually constant, thus it does not converge by discreteness. Hence, (X, d) is not metrically complete. To prove (ii)$\Rightarrow$(iii) suppose that there is a distance ball that is infinite. By Lemma 4.5 there is a bounded infinite geodesic and (X, d) is not geodesically complete. From Lemma 4.4 (b) we deduce (iii)⇔(iv). Finally, we consider the direction (iv)⇒(i). If every bounded and closed set is compact, then every closed distance ball is compact. Then, by Lemma 4.4 (b) every distance ball is finite and it follows that (X, d) is metrically complete. □

4.4. **Some important conditions.** In the general situation when one is not necessarily locally finite or has a path metric, we will often make assumptions. These assumptions are discussed next.

We say a pseudo metric ρ admits *finite balls* if (iii) in the theorem above is satisfied for ρ, i.e.,

(B) The distance balls $B_r(x) = \{y \in X \mid \rho(x, y) \leq r\}$ are finite for all $x \in X$, $r \geq 0$.

A somewhat weaker assumption is that the *degree is bounded on balls*:

(D) The restriction of Deg to $B_r(x)$ is bounded for all $x \in X$, $r \geq 0$.

Clearly, (B) implies (D). Moreover, (D) is equivalent to the fact that $\mathcal{L}$ restricted to the ℓ^2 space of a distance ball is a bounded operator.

The assumptions (B) and (D) can be understood as bounding ρ in a certain sense from below. Next, we come to an assumption which may be understood as an upper bound.

We say a pseudo-metric ρ has *finite jump size* if

(J) The jump size $s = \sup\{\rho(x, y) \mid x, y \in X,\ x \sim y\}$ is finite.

The assumptions (B) and (J) combined have strong geometric consequences.

Lemma 4.6. *Let (b, c) be a graph over (X, m) and ρ be an pseudo metric. If ρ satisfies (B) and (J), then the graph is locally finite.*

Proof. If there was a vertex with infinitely many neighbors, then there would be a distance ball containing all of them by finite jump size. However, this is impossible by (B). □

4.5. **Construction of cut-off functions.** From an analyst's point of view a major interest in metrics is to construct cut-off functions with desirable properties. Let us illustrate in which sense intrinsic metrics serve this purpose.

Given a intrinsic metric ρ, a subset $A \subseteq X$ and $R > 0$, the most basic cut-off function is defined by $\eta = \eta_{A,R} : X \to [0, \infty)$

$$\eta(x) = \Big(1 - \frac{\rho(x, A)}{R}\Big) \wedge 0, \qquad x \in X.$$

Such test functions are often used to approximate a solution $f \in C(X)$ by $\eta_{B_r,R} f$, where B_r is a ball with respect to ρ about some vertex.

If (B) holds, then $\eta_{B_r,R} f$ is in $C_c(X)$ which is for example sufficient to apply Green's formula.

Secondly, if (J) holds, i.e., $s < \infty$. Then, the function $d\eta$ on the edges given by

$$d\eta(x, y) = \eta(x) - \eta(y)$$

is supported on $B_{R+s} \setminus B_{r-s}$.

Let us collect some basic properties.

Lemma 4.7. *Let $\eta = \eta_{B_r,R}$, $0 < r < R$, be given as above. Then,*

(a) $\eta|_{B_r} \equiv 1$ *and* $\eta|_{X\setminus B_R} \equiv 0$.

(b) *For $x \in X$,*

$$\sum_{y\in X} b(x, y)|\eta(x) - \eta(y)|^2 \leq \frac{1}{(R-r)^2} 1_{B_{R+s}\setminus B_{r-s}}(x)m(x).$$

Proof. (a) is obvious from the definition of η and (b) follows directly from

$$\begin{aligned}\sum_{y\in X} b(x, y)(\eta(x)-\eta(y))^2 &= \sum_{y\in B_{R+s}\setminus B_{r-s}} b(x, y)(\eta(x) - \eta(y))^2 \\ &\leq \frac{1}{(R-r)^2} \sum_{y\in B_{R+s}\setminus B_{r-s}} b(x, y)(\rho(x, B_r) - \rho(y, B_r))^2 \\ &\leq \frac{1}{(R-r)^2} \sum_{y\in B_{R+s}\setminus B_{r-s}} b(x, y)\rho^2(x, y) \\ &\leq \frac{1}{(R-r)^2} m(x),\end{aligned}$$

for $x \in B_{R+s} \setminus B_{r-s}$ and the term is 0 otherwise. □

5. Liouville type theorems

The classical Liouville theorem states that if a harmonic function is bounded from below, then the function is constant. Here, we look into boundedness assumptions such as ℓ^p growth bounds. First, we present Yau's L^p Liouville theorem and Karp's improved bound in the case of manifolds. Secondly, we discuss the case of the normalized Laplacian for graphs and the results that have been proven for this operator. Thirdly, we present theorems that recover Yau's and Karp's results for weighted graphs using intrinsic metrics. This yields a sufficient criterion for recurrence. Finally, we round off the section by a result of a type which seems to hold exclusively for discrete spaces.

5.1. **Historical notes on Yau's and Karp's theorem.** We consider a connected Riemannian manifold M, together with its Laplace Beltrami operator Δ_M. A twice continuously differentiable function f on M is called harmonic (respectively subharmonic) if $\Delta_M f = 0$ (respectively $\Delta_M f \leq 0$).

In 1976 Yau [47] proved that on a complete Riemannian manifold M any harmonic function or positive subharmonic function in $L^p(M)$ is already constant.

This result was later strengthened by Karp in 1982. Namely, any harmonic function or positive subharmonic function f that satisfies

$$\inf_{r_0>0}\int_{r_0}^{\infty}\frac{1}{\|f1_{B_r}\|_p^p}dr=\infty,$$

is already constant, where 1_{B_r} is the characteristic function of the geodesic ball B_r about some arbitrary point in the manifold. Karp's result has Yau's theorem as a direct consequence.

Later in 1994 Sturm generalized Karp's theorem to strongly local Dirichlet forms, where balls are taken with respect to the intrinsic metric. The underlying assumption on the metric is that it generates the original topology and all balls are relatively compact.

For graphs b over (X, m) the first results in this direction were obtained for the normalizing measure $m = n$. In this case, the operator L is bounded, see Section 2.5, and the combinatorial graph distance d is an intrinsic metric, see Section 4.2.2. (Of course, the fact that a function is harmonic depends only on the graph b and not on the measure m, but being in an ℓ^p space does.)

Starting 1997 with Holopainen/Soardi [18], Rigoli/Salvatori/Vignati [38], Masamune [33], eventually in 2013 Hua/Jost [19] showed that if a harmonic or positive subharmonic function f satisfies

$$\liminf_{r\to\infty}\frac{1}{r^2}\|f1_{B_r(x)}\|_p^p<\infty,$$

for some $p \in (1,\infty)$ and $x \in X$, then f must be constant. Here, the balls are taken with respect to the natural graph distance. This directly implies Yau's theorem for $p \in (1,\infty)$. Moreover, Hua/Jost [19] also show Yau's theorem for $p = 1$.

5.2. **Liouville theorems and intrinsic metrics.** For graphs b over a general discrete measure space one can not expect such results to hold without any further conditions. Since the property of being harmonic does not depend on the measure, for any harmonic function f there is a measure m such that f is in $\ell^p(X, m)$, $p \in (1,\infty)$.

The following theorem for $p \in (1,\infty)$ is found in [20, Corollary 1.2] with the additional assumption of finite jump size (J). Below we sketch how to omit (J) an idea which was communicated by Huang [22]. For

$p = 2$, the theorem (without the assumption (J)) is found in [14] based on ideas developed in [20, 24] and [36].

Theorem 5.1 (Corollary 1.2 in [20] and [22])**.** *Let (b, c) be a connected graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D). If $f \in \ell^p(X, m)$, $p \in (1, \infty)$, is a positive subharmonic function then f is constant.*

Refining the Caccioppoli inequality from the proof of Theorem 5.1 above, see [20, Lemma 3.1], and applying it recursively, we obtain a discrete version of Karp's theorem. However, for the recursive scheme we need the finite jump size assumption (J).

Theorem 5.2 (Theorem 1.1 in [20])**.** *Let b be a connected graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). If f is a positive subharmonic function such that for some $p \in (1, \infty)$ and $x \in X$*

$$\inf_{r_0>0} \int_{r_0}^{\infty} \frac{1}{\|f 1_{B_r(x)}\|_p^p} dr = \infty,$$

then f is constant, where 1_B is the characteristic function of a set $B \subseteq X$.

In particular, the theorem above implies the result of Hua/Jost [19] and even implies that a harmonic function f satisfying

$$\limsup_{r\to\infty} \frac{1}{r^2 \log r} \|f 1_{B_r(x)}\|_p^p < \infty,$$

for some $p \in (1, \infty)$, is constant.

5.3. **Green's formula, Leibniz rules and mean value theorem.** We first prove a Green's formula which is an L^p version of the one in [24, Lemma3.3]. The improvement is an idea due to Huang [22]. For a set U let $n(U)$ be the combinatorial neighborhood of U, that is

$$n(U) = \{y \in X \mid \text{ there is } x \in U\ x \sim y\} \cup U.$$

Lemma 5.3 (Green's formula)**.** *Let (b, c) be a connected graph over (X, m) and ρ an intrinsic metric. Let $p, q \in [1, \infty)$ such that $p^{-1} + q^{-1} = 1$, $\varepsilon > 0$ and assume* Deg *is bounded on $B_\varepsilon(U)$. Then for all f with $f 1_{n(U)} \in \ell^p(X, m) \cap \mathcal{F}$ and $g \in \ell^q(X, m)$ with $\operatorname{supp} g \subseteq U$*

$$\sum_{x\in X} (\mathcal{L}f)(x)\overline{g(x)}m(x) = \frac{1}{2} \sum_{x,y\in U} b(x, y)(f(x) - f(y))\overline{(g(x) - g(y))}.$$

Proof. We let $b' = b - b1_{X\setminus U\times X\setminus U}$ and $c' = c1_U$. Denote by $\mathcal{L}'$ the formal Laplacian for the graph (b', c') and Deg' the corresponding weighted degree. We claim that Deg' is bounded on $n(U)$. Clearly, $\text{Deg}' = \text{Deg}$

on U and $\mathrm{Deg}' \leq \mathrm{Deg}$ on $B_\varepsilon(U)$, therefore, it is bounded by assumption. For $x \in n(U) \setminus B_\varepsilon(U)$ note that by the property of ρ

$$\begin{aligned}\varepsilon^2 \mathrm{Deg}'(x) &\leq \frac{\varepsilon^2}{m(x)} \sum_{y\in X} b'(x,y) = \frac{\varepsilon^2}{m(x)} \sum_{y\in U} b(x,y) \\ &\leq \frac{1}{m(x)} \sum_{y\in U} b(x,y)\rho^2(x,y) \leq 1\end{aligned}$$

and, hence, $\mathrm{Deg}'(x) \leq \varepsilon^{-2}$ for $x \in n(U) \setminus B_\varepsilon(U)$. Now, as Deg' is bounded on $n(U)$ we get by Hölder's inequality

$$\Big| \sum_{x,y\in n(U)} b'(x,y) f(x)\overline{g(y)} \Big| \leq \Big(\sum_{n(U)} |f|^p \mathrm{Deg}'\, m\Big)^{\frac{1}{p}} \Big(\sum_{n(U)} |g|^q \mathrm{Deg}'\, m\Big)^{\frac{1}{q}} < \infty,$$

where the finiteness follows from the boundedness of Deg', $f 1_{n(U)} \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$. Simultaneously,

$$\Big| \sum_{x,y\in n(U)} b'(x,y) f(x)\overline{g(x)} \Big| \leq \Big(\sum_{n(U)} |f|^p \mathrm{Deg}'\, m\Big)^{\frac{1}{p}} \Big(\sum_{n(U)} |g|^q \mathrm{Deg}'\, m\Big)^{\frac{1}{q}} < \infty.$$

Now, we proceed as in the proof of [24, Lemma3.3]. Note that absolute convergence of every term is ensured by the estimates above

$$\begin{aligned}&\sum_{x\in X} (\mathcal{L} f)(x)\overline{g(x)} m(x) \\ &\quad= \sum_{x\in U} (\mathcal{L}' f)(x)\overline{g(x)} m(x) \\ &\quad= \sum_{x\in U, y\in n(U)} b'(x,y) df(x,y)\overline{g(x)} + \sum_{x\in U} c'(x) f(x)\overline{g(x)} \\ &\quad= \sum_{x,y\in n(U)} b'(x,y) df(x,y)\overline{g(x)} + \sum_{x\in n(U)} c'(x) f(x)\overline{g(x)} \\ &\quad= \frac{1}{2} \sum_{x,y\in n(U)} b'(x,y) df(x,y)\overline{dg(x,y)} + \sum_{x\in n(U)} c'(x) f(x)\overline{g(x)} \\ &\quad= \frac{1}{2} \sum_{x,y\in X} b(x,y) df(x,y)\overline{dg(x,y)} + \sum_{x\in X} c(x) f(x)\overline{g(x)}.\end{aligned}$$

□

The following Leibniz rules follow by direct computations.

Lemma 5.4 (Leibniz rules). *For all $x,y \in X$, $x \sim y$ and $f,g : X \to \mathbb{R}$*

$$\begin{aligned}d(fg)(x,y) &= f(y)dg(x,y) + g(x)df(x,y) \\ &= f(y)dg(x,y) + g(y)df(x,y) + df(x,y)dg(x,y).\end{aligned}$$

A fundamental difference of Laplacians on graphs and on manifolds is the absence of a chain rule in the graph case. In particular, existence of a chain rule can be used as a characterization for a regular Dirichlet form to be strongly local. We circumvent this problem by using the mean value theorem from calculus instead. In particular, for a continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ and $f : X \to \mathbb{R}$, we have

$$d(\varphi \circ f)(x, y) = \varphi'(\zeta)(f(x) - f(y)),$$

for some $\zeta \in [f(x) \wedge f(y), f(x) \vee f(y)]$. In this paper we will apply this formula to get estimates for the function $\varphi : t \mapsto t^{p-1}$, $p \in (1, \infty)$. However, we need a refined inequality as it was already used in the proof of [18, Theorem 2.1]. For the convenience of the reader, we include a short proof here.

Lemma 5.5 (Mean value inequalities). *For all $f : X \to \mathbb{R}$ and $x \sim y$ with $(f(x) - f(y)) \geq 0$,*

(a) $(f(x) - f(y))^{p-1} \geq \frac{1}{2}(f^{p-2}(x) + f^{p-2}(y))(f(x) - f(y))$, *for* $p \in [2, \infty)$,

(b) $(f(x) - f(y))^{p-1} \geq C(f(x) \vee f(y))^{p-2}(f(x) - f(y))$, *for* $p \in (1, \infty)$, *where* $C = (p - 1) \wedge 1$.

Proof. (a) Denote $a = f(y)$, $b = f(x)$. As it is the only non-trivial case, we assume $0 < a < b$. Note that for $p \neq 1$

$$b^{p-1} - a^{p-1} = (b - a)(b^{p-2} + a^{p-2}) + ab(b^{p-3} - a^{p-3}).$$

Thus, the statement is immediate for $p \geq 3$ since the second term on the right side is non-negative in this case. Let $2 \leq p < 3$ and note $a^{p-3} > b^{p-3}$. The function $t \mapsto t^{2-p}$ is convex on $(0, \infty)$ and, thus, its image lies below the line segment connecting (b^{-1}, b^{p-2}) and (a^{-1}, a^{p-2}). Therefore,

$$\begin{aligned} a^{p-3} - b^{p-3} &\leq \frac{a^{p-3} - b^{p-3}}{(3 - p)} \\ &= \int_{b^{-1}}^{a^{-1}} t^{2-p} dt \leq (a^{-1} - b^{-1})\Big(\frac{(b^{p-2} - a^{p-2})}{2} + a^{p-2}\Big) \\ &= \frac{1}{2ab}(b - a)(a^{p-2} + b^{p-2}). \end{aligned}$$

From the equality in the beginning of the proof we now deduce the assertion in the case $2 \leq p < 3$.
(b) The case $p \geq 2$ follows from (a). The case $1 < p \leq 2$ in (b) follows directly from the mean value theorem. □

5.4. **The key estimate.** The lemma below is vital for the proof of Yau's and Karp's theorem.

Lemma 5.6 (Lemma 3.1 in [20])**.** *Let* $p \in (1,\infty)$*,* $0 \le \varphi \in \ell^\infty(X)$*,* $\varepsilon > 0$ *and* $U = \operatorname{supp}\varphi$*. Assume* Deg *is bounded on* $B_\varepsilon(U)$*. Then, for every non-negative subharmonic function* f *with* $f1_{n(U)} \in \ell^p(X,m)$*,*

$$\sum_{x,y\in X} b(x,y)(f(x)\vee f(y))^{p-2}\varphi(y)^2|df(x,y)|^2$$
$$\le C \sum_{x,y\in n(U)} b(x,y)f^{p-1}(x)\varphi(y)|df(x,y)d\varphi(x,y)|,$$

where $C = 4/((p-1)\wedge 1)$*.*

Proof. From the assumptions $f1_U \in \ell^p(X,m)$ and $\varphi \in \ell^\infty(X)$, we infer $\varphi^2 f^{p-1} \in \ell^q(X,m)$ with $q = p/(p-1)$, i.e., $p^{-1}+q^{-1} = 1$. Thus, boundedness of Deg on U implies applicability of Green's formula with f and $g = \varphi^2 f^{p-1}$. We start by using non-negativity and subharmonicity of f then applying Green's formula (Lemma 5.3) and the first and second Leibniz rule (Lemma 5.4)

$$\begin{aligned}0 \ge& \sum_{x\in X}(\mathcal{L}f)(x)(\varphi^2 f^{p-1})(x)m(x)\\ =&\frac{1}{2}\sum_{x,y\in X} b(x,y)df(x,y)d(\varphi^2 f^{p-1})(x,y)\\ =&\frac{1}{2}\sum_{x,y\in X} b(x,y)df(x,y)\big(\varphi^2(y)df^{p-1}(x,y) + f^{p-1}(x)d\varphi^2(x,y)\big)\\ =&\frac{1}{2}\sum_{x,y\in X} b(x,y)df(x,y)\\ &\qquad\cdot\big(\varphi^2(y)df^{p-1}(x,y) + 2f^{p-1}(x)\varphi(y)d\varphi(x,y) + f^{p-1}(x)|d\varphi(x,y)|^2\big)\end{aligned}$$

We apply the mean value inequality, Lemma 5.5 (b), to the first term, that is

$$\varphi^2(y)df(x,y)df^{p-1}(x,y) \ge C\varphi^2(y)(f^{p-1}(x)\vee f^{p-1}(y))^{p-2}|df(x,y)|^2$$

and for third term we notice that

$$\begin{aligned}&\sum_{x,y\in X} b(x,y)f^{p-1}(x)df(x,y)|d\varphi(x,y)|^2\\ &\qquad= \frac{1}{2}\sum_{x,y\in X} b(x,y)df^{p-1}(x,y)df(x,y)|d\varphi(x,y)|^2 \ge 0\end{aligned}$$

Hence, we obtain

$$\begin{aligned}0 \ge&\frac{C}{2}\Big(\sum_{x,y\in X} b(x,y)\Big(\varphi^2(y)(f^{p-1}(x)\vee f^{p-1}(y))^{p-2}|df(x,y)|^2\\ &+ 2\sum_{x,y\in X} b(x,y)f^{p-1}(x)\varphi(y)df(x,y)d\varphi(x,y)\Big)\end{aligned}$$

Absolute convergence of the two terms in the last line can be checked using Hölder's inequality and the assumptions $f1_U \in \ell^p(X,m)$, $\varphi \in \ell^\infty(X)$ and boundedness of Deg on U. Hence, we obtain the statement of the lemma. □

5.5. **Caccioppoli inequality and proof of Yau's theorem.**

Theorem 5.7 (Caccioppoli-type inequality, Theorem 1.8 in [20])**.** *Let (b,c) be a connected graph over (X,m) and ρ be an intrinsic metric with bounded degree on balls (D). Let $p \in (1,\infty)$. Then, there is $C > 0$ such that for every non-negative subharmonic function f and all $0 < r < R$*

$$\sum_{x,y\in B_r} b(x,y)(f(x)\vee f(y))^{p-2}|(f(x)-f(y))|^2 \leq \frac{C}{(R-r)^2}\|f1_{B_{R+s}\setminus B_{r-s}}\|_p^p,$$

where s is the jump size of the intrinsic metric.

Proof. Using Lemma 5.6 and the inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, $\varepsilon > 0$, we estimate

$$\begin{aligned}
&\sum_{x,y\in B_r} b(x,y)(f(x)\vee f(y))^{p-2}\varphi(y)^2|df(x,y)|^2\\
&\leq C\sum_{x,y\in n(B_r)} b(x,y)f^{p-1}(x)\varphi(y)|df(x,y)d\varphi(x,y)|\\
&\leq \frac{1}{2}\sum_{x,y\in X)} b(x,y)(f(x)\vee f(y))^{p-2}\varphi(y)^2|df(x,y)|^2\\
&\quad + C\sum_{x,y\in X} b(x,y)(f(x)\vee f(y))^{p}|d\varphi(x,y)|^2.
\end{aligned}$$

Letting $\varphi = \eta = \eta_{B_r,R}$ with $0 < r < R$ and

$$\eta(x) = \Big(1 - \frac{\rho(x,B_r)}{R}\Big)\wedge 0, \qquad x \in X.$$

and using the cut-off function lemma, Lemma 4.7, we arrive at

$$\begin{aligned}
&\sum_{x,y\in B_r} b(x,y)(f(x)\vee f(y))^{p-2}\eta(y)^2|df(x,y)|^2\\
&\leq C\sum_{x,y\in B_r} b(x,y)(f(x)\vee f(y))^{p}|d\varphi(x,y)|^2 \leq \frac{C}{(R-r)^2}\|f1_{B_{R+s}\setminus B_{r-s}}\|_p^p.
\end{aligned}$$

□

Proof of Theorem 5.1. Letting first $R \to \infty$ and then $r \to \infty$ in the Caccioppoli inequality, we see as $f \in \ell^p(X,m)$, for all x

$$(f(x)\vee f(y))^{p-2}|(f(x)-f(y))|^2 = 0$$

for all $x \sim y$ which immediately implies that f is constant. □

5.6. **Proof of Karp's theorem.**

Proof of Theorem 5.2. Let $p \in (1,\infty)$ and let f be a non-negative subharmonic function. Assume $f1_{B_r} \in \ell^p(X,m)$ for all $r \geq 0$ since otherwise $\inf_{r_0}\int_{r_0}^{\infty} r/\|f1_{B_r}\|_p^p dr = 0$. We assume finite jump size $s < \infty$. Let $\eta = \eta_{B_{r+s},R-s}$ with $0 < r < R-3s$ and Letting $\varphi = \eta = \eta_{r+s,R-s}$ with $0 < r < R-3s$ and

$$\eta(x) = \Big(1 - \frac{\rho(x,B_{r+s})}{R-s}\Big) \wedge 0, \qquad x \in X.$$

Then by Lemma 5.6 (applied with $\varphi = \eta$) we obtain

$$\begin{aligned}
&\sum_{x,y\in B_R} b(x,y)(f(x)\vee f(y))^{p-2}\eta(y)^2|df(x,y)|^2\\
&= \sum_{x,y\in B_{r+2s}} b(x,y)(f(x)\vee f(y))^{p-2}\eta(y)^2|df(x,y)|^2\\
&\leq C \sum_{x,y\in n(B_{r+2s})} b(x,y)f^{p-1}(x)\varphi(y)|df(x,y)d\varphi(x,y)|\\
&= C \sum_{x,y\in B_R\setminus} b(x,y)f^{p-1}(x)\varphi(y)|df(x,y)d\varphi(x,y)|
\end{aligned}$$

where we use $r < R-3s$ and $d\eta(x,y) = 0$, $x,y \in B_r$ in the last line. Now, the Cauchy-Schwarz inequality and the cut-off function lemma, Lemma 4.7, yield

$$\begin{aligned}
&\Big(\sum_{x,y\in B_R} b(x,y)(f(x)\vee f(y))^{p-2}\eta(y)^2|df(x,y)|^2\Big)^2\\
&\leq C\Big(\sum_{x,y\in B_R\setminus B_r} b(x,y)f^p(x)|d\varphi(x,y)|^2\Big)\Big(\sum_{x,y\in B_R\setminus B_r} b(x,y)f^{p-2}(x)\varphi(y)^2|df(x,y)|^2\Big)\\
&\leq \frac{C}{(R-r)^2}\|f1_{B_R\setminus B_r}\|_p^p \left(\Big(\sum_{x,y\in B_r} - \sum_{x,y\in B_r}\Big) b(x,y)f^{p-2}(x)\varphi(y)^2|df(x,y)|^2\right.
\end{aligned}$$

Let $R_0 \geq 3s$ be such that $f1_{B_{R_0}} \neq 0$ and denote

$$v(r) = \|f1_{B_r}\|_p^p, \qquad r \geq 0.$$

Moreover, for $j \geq 0$, let $R_j = 2^j R_0$, $\varphi_j = \eta_{R_j+s,R_{j+1}-s}$ and

$$Q_{j+1} = \sum_{x,y\in B_{R_{j+1}}} b(x,y)f^{p-2}(x)\varphi_j^2(y)|df(x,y)|^2.$$

As $\varphi_{j-1} \leq \varphi_j$, we get $Q_j \leq Q_{j+1}$ and together with the estimate above this implies

$$Q_jQ_{j+1} \leq Q_{j+1}^2 \leq C\frac{v(R_{j+1})}{(R_{j+1}-R_j)^2}(Q_{j+1}-Q_j), \qquad j \geq 0.$$

Since $R_{j+1} = 2R_j$, dividing the above inequality by $\frac{v(R_{j+1})}{R_{j+1}^2} Q_j Q_{j+1}$ and adding C/Q_{j+1} yield

$$\frac{R_{j+1}^2}{v(R_{j+1})} + \frac{C}{Q_{j+1}} \leq \frac{C}{Q_j}$$

and, thus,

$$\frac{1}{C} \sum_{j=1}^{\infty} \frac{R_{j+1}^2}{v(R_{j+1})} \leq \frac{1}{Q_1}.$$

Now, the assumption $\int_{R_0}^{\infty} r/v(r) dr = \infty$ implies $\sum_{j=0}^{\infty} \frac{R_j^2}{v(R_j)} = \infty$. Therefore, $Q_1 = 0$. As this is true for all R_0 large enough, we have

$$(f(x) \vee f(y))^{p-2} |df(x,y)|^2 = 0,$$

for all edges $x \sim y$. For $p \geq 2$, connectedness clearly implies that f is constant. On the other hand, for $p \in (1, 2]$, we always have $f^{p-2}(e_+) > 0$ and, thus, f is constant. □

5.7. **Domain of the generator.** In Lemma 2.6 we demonstrated that for a graph we may not have $\mathcal{L} C_c(X) \subseteq \ell^2(X, m)$. Hence, in general $\mathcal{L}$ is not a symmetric operator on the subspace $C_c(X)$ of $\ell^2(X, m)$. Nevertheless, we can still specify the domain of the operator. The following result is found in [24] for graph Laplacians and in [14] for magnetic Schrödinger operators. We only sketch the idea of the proof which follows essentially from Theorem 5.1 above and standard arguments found in [28, Proof of Theorem 5 and 6]. One may find a version of the Theorem below in [20, Corollary 1.4].

Theorem 5.8. *Let (b, c) be a graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D). Then,*

$$D(L_p) = \{f \in \ell^p(X, m) \mid \mathcal{L} f \in \ell^p(X, m)\}, \quad \textit{for all } p \in (1, \infty).$$

In particular, if additionally $\mathcal{L} C_c(X) \subseteq \ell^2(X, m)$, then $\mathcal{L}|_{C_c(X)}$ is essentially selfadjoint on $\ell^2(X, m)$.

Idea of the proof for $p = 2$. For two extensions L' and L'' one sees by Green's formula that both are restrictions of $\mathcal{L}$. Then $u = (L' - \lambda)^{-1} f - (L'' - \lambda)^{-1} f$ is a solution to $(\mathcal{L} - \lambda) u = 0$ in $\ell^p(X, m)$ for any $f \in \ell^2(X, m)$. By Theorem 5.1 and Lemma 2.1 we know $u \equiv 0$. Hence, $(L' - \lambda)^{-1} = (L'' - \lambda)^{-1}$ and, therefore, $L' = L''$. □

Proof of Theorem 5.8 (Domain of the ℓ^p generators). Let $f \in \ell^p(X, m) \cap \mathcal{F}$ be such that $(\mathcal{L} + 1) f = 0$. Since the positive and negative part f_+, f_- of f are non-negative, subharmonic and in $\ell^p(X, m)$, they must be constant by Yau's theorem, Theorem 5.1. This implies $f_\pm \equiv 0$ and, thus, $f \equiv 0$. Now, the proof of the corollary works literally line by line as the proof of Theorem 3.2. □

Combining this with the Hopf-Rinow type theorem, Theorem 4.3, above, we obtain an analogue to a classical result in Riemannian geometry which as discussed above is often referred to as Gaffney's theorem.

Corollary 5.9 (Theorem 2 in [24]). *Let b be a locally finite graph over (X, m) and ρ be an intrinsic path metric. If (X, ρ) is metrically complete, then $\mathcal{L}|_{C_c(X)}$ is essentially selfadjoint on $\ell^2(X, m)$.*

5.8. **Recurrence.** As a direct consequence of Karp's theorem we get a sufficient criterion for recurrence of a graph. A connected graph is called *recurrent* if for all m and some (all) $x, y \in X$, we have

$$\int_0^\infty e^{-tL} 1_{\{x\}}(y)dt = \infty$$

which is equivalent to absence of non-constant bounded subharmonic functions. For a collection of various equivalent statements of recurrence, see [20, Proposition 3.3] and references therein.

Similar analogous results to the criterion below are due to [25, Theorem 3.5] and [43, Theorem 3] which generalizes for example [3, Theorem 2.2], [38, Corollary B], [45, Lemma 3.12], [12, Corollary 1.4], [35, Theorem 1.2] on graphs.

Theorem 5.10 (Corollary 1.6 in [20]). *Let b be a connected graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). If for some $x \in X$*

$$\int_1^\infty \frac{r}{m(B_r(x))} dr = \infty,$$

then the graph is recurrent.

For the proof of the theorem above we recall the following well known equivalent conditions for recurrence.

Proposition 5.11 (Characterization of recurrence). *Let a connected graph X be given. Then the following are equivalent.*

(i) *For the transition matrix P with $P_{x,y} = b(x, y)/\sum_{z\in X} \mu_{xz}$, $x, y \in X$, we have $\sum_{n=0}^\infty P^{(n)}(x, y) = \infty$ for some (all) $x, y \in X$, where $P^{(n)}$ denotes the n-th power of P.*

(ii) *For $m \equiv 1$ and some (all) $x, y \in X$, we have $\int_0^\infty e^{-tL}\delta_x(y)dt = \infty$, where $\delta_x(y) = 1$ if $x = y$ and zero otherwise.*

(iii) *For all m and some (all) $x, y \in X$, we have $\int_0^\infty e^{-tL}\delta_x(y)dt = \infty$.*

(iv) *Every bounded superharmonic (or subharmonic) function is constant.*

(v) *Every non-negative superharmonic function is constant.*

(vi) *Every superharmonic (or subharmonic) function of finite energy is constant.*

(vii) $\mathrm{cap}(x) := \inf\{E(f) | f \in C_c(X), f(x) = 1\} = 0$ *for some (all)* $x \in X$

A graph is called *recurrent* if one of the equivalent statements of Proposition 5.11 is satisfied.

Proof. The equivalence (i)⇔(ii) is shown in [39, Theorem 6]. The equivalences (ii)⇔(vi)⇔(iii) are in [39, Theorem 2 and Theorem 9] (confer [41, Theorem 3.34]). The equivalences (i)⇔(v)⇔(vii) are found in [45, Theorem 1.16, Theorem 2.12]. The equivalence (iv)⇔(v) follows since every non-negative superharmonic function f can be approximated by the bounded superharmonic functions $f \wedge n$, $n \geq 1$. □

Proof of Theorem 5.10 (Recurrence). Theorem 5.2 implies that any non-negative bounded subharmonic function f is constant provided $\inf_{r_0} \int_{r_0}^{\infty} r/m(B_r)dr = \infty$ since $\|f1_{B_r}\|_p^p \leq \|f\|_\infty^p m(B_r)$, $r \geq 0$. By Proposition 5.11 the graph is recurrent. □

5.9. **L^1-Liouville theorem and counter-examples.** In this section we deal with the borderline case of the ℓ^p Liouville theorem $p = 1$. We first prove Theorem 5.12 which deals with the stochastic complete case and then give two examples which show that there is no ℓ^1 Liouville theorem for non-negative subharmonic functions in the general case.

A graph is called *stochastically complete* if $e^{-tL}1 = 1$, where 1 denotes the function that is constantly one on X. For the relevance of the concept see [11, 28, 46]. The proof of Theorem 5.12 follows along the lines of the proof of [11, Theorem 13.2].

Theorem 5.12 (Grigor'yan's L^1 theorem)**.** *Assume a connected graph is stochastically complete. Then, every non-negative superharmonic function in $\ell^1(X, m)$ is constant.*

Proof of Theorem 5.12. If the graph is recurrent, then there are no non-constant non-negative superharmonic functions by Proposition 5.11. So assume the graph is not recurrent which implies $G(x, y) = \int_0^\infty e^{-tL}\delta_x(y)dt < \infty$, $x, y \in X$, again by Proposition 5.11. Let K_n, $n \geq 0$, be an sequence of finite sets exhausting X and $G_n(x, y) = \int_0^\infty e^{-tL_n}\delta_x(y)dt$, where L_n are the finite dimensional operators arising from the restriction of the form Q to $C_c(K_n)$. By domain monotonicity, [28, Proposition 2.6 and 2.7] the semigroups e^{-tL_n} converge monotonously increasing to e^{-tL} and, hence, $G_n(x, y) \leq G(x, y)$ for $x, y \in K_n$, and $G_n \nearrow G$, $n \to \infty$, pointwise. By direct calculation for any $x \in K_n$

$$L_n G_n(x, y) = \int_0^\infty L_n e^{-tL_n}\delta_x(y)dt = \int_0^\infty \partial_t e^{-tL_n}\delta_x(y)dt$$
$$= [e^{-tL_n}\delta_x(y)]_0^\infty = \delta_x(y)$$

and, hence, $G_n(x, \cdot)$ are harmonic on $K_n \setminus \{x\}$, $n \geq 0$.

Let u be a non-trivial non-negative superharmonic function which is strictly positive by the minimum principle [28, Theorem 8]. Let $U \subseteq X$ be finite with $o \in U \subseteq K_n$, $n \geq 0$ and $C > 0$ be such that $Cu \geq G(o, \cdot)$ on U. By the minimum principle $Cu \geq G_n(o, \cdot)$ on $K_n \setminus \{o\}$ and, hence, $Cu \geq G(o, \cdot)$ on X by the discussion above. If the graph is stochastically complete, then we get by Fubini's theorem

$$C\|u\|_1 \geq \|G(o,\cdot)\|_1 = \int_0^\infty \sum_{x\in X} e^{-tL}\delta_o(x)m(x)dt = \int_0^\infty e^{-tL}1(o)dt$$
$$= \int_0^\infty dt = \infty.$$

Hence, u is not in $\ell^1(X, m)$. □

In the proof we show that in the non-recurrent case there are no nontrivial superharmonic functions in ℓ^1. This is explained since in the case of finite measure recurrence and stochastic completeness are equivalent [39, Theorem 12].

Next, we show that in general there is no ℓ^p Liouville theorem for $p \in (0, 1]$. This is analogous to the situation in Riemannian geometry, where counter-examples were given by Chung and Li/Schoen. Our first example is a graph of finite volume and the second is of infinite volume.

Example 5.13 (Finite volume)**.** Let $G = (X, \mu, m)$ be an infinite line graph, i.e., $X = \mathbb{Z}$ and $x \sim y$ iff $|x - y| = 1$ for $x, y \in \mathbb{Z}$. Define the edge weight by $b(x, y) = 2^{1-(|x|\vee|y|)}$ for $x \sim y$ and the measure m by $m(x) = (|x| + 1)^{-2}2^{-|x|}$, $x \in \mathbb{Z}$, which implies $m(X) < \infty$. The vertex degree path metric ρ_0 is compatible as it satisfies $\delta(x, x + 1) \geq C(|x|+1)^{-1}$ and, thus, $\sum_{x=-\infty}^{\infty} \delta(x, x+1) = \infty$. However, the function f defined as

$$f(x) = \operatorname{sign}(x)(2^{|x|} - 1), \quad x \in \mathbb{Z},$$

is harmonic and, clearly, $f \in \ell^p(X, m)$, $p \in (0, 1]$.

Example 5.14 (Infinite volume)**.** We can extend the example above to the infinite volume case. Let G be the graph from above and G' be a locally finite graph of infinite volume which allows for a compatible path metric. We glue G' to the vertex $x = 0$ of the graph G by identifying a vertex in G' with $x = 0$. Next, we extend the path metrics in the natural way and obtain (by renormalizing the edge weights of the metric at the edges around $x = 0$ if necessary) again a compatible intrinsic metric and the graph has infinite volume. Moreover, we extend f on G from above by zero to G' and obtain a harmonic function which is in ℓ^p, $p \in (0, 1]$.

6. Independence of the spectrum

In the beginning of the 80's Simon [40] asked a famous question whether the spectra of certain Schrödinger operators on $\mathbb{R}^d$ are independent on which L^p space they are considered. Hempel/Voigt [17] gave an affirmative answer in 1986. Here, we consider a geometric analogue of this question. First, we discuss a theorem by Sturm on Riemannian manifolds and secondly, we present a result for weighted graphs involving intrinsic metrics.

6.1. **Historical notes.** In 1993 Sturm proved a theorem for uniformly elliptic operators on a complete Riemannian manifold M whose Ricci curvature is bounded below. We assume that M grows uniformly subexponentially if for any $\varepsilon > 0$ there is $C > 0$ such that for all $r > 0$ and all $x \in M$

$$\mathrm{vol}(B_r(x)) \leq Ce^{\varepsilon r}\mathrm{vol}(B_1(x)).$$

Then the spectrum of a uniformly elliptic operator on this manifold is independent of the space $L^p(M)$, $p \in [1, \infty]$ on which it is considered.

6.2. **The result for graphs with intrinsic metrics.** A graph (b, c) over (X, m) with an intrinsic metric ρ is said to have *uniform subexponential growth* if for any $\varepsilon > 0$ there is $C > 0$ such that for all $r > 0$ and all $x \in M$

$$m(B_r(x)) \leq Ce^{\varepsilon r}m(x).$$

The proof of the following theorem follows closely the strategy of Sturm in [42].

Theorem 6.1 (Theorem 1 in [1]). *Let b be a connected graph over (X, m) and ρ be an intrinsic metric such that the balls are finite (B), which has finite jump size (J) and the graph grows uniformly subexponentially. Then,*

$$\sigma(L_p) = \sigma(L_2), \qquad p \in [1, \infty].$$

The statement of the theorem is in general wrong if one drops the growth assumption. In particular, if $\lambda_0(L_2) > 0$, then the graph grows exponentially, i.e., $\mu > 0$ by the section above. On the other hand, if the graph is stochastically complete, then 1 is an eigenfunction of L_∞ to the eigenvalue 0 and, therefore, $\lambda_0(L_1) = 0$ by duality. Hence, $\lambda_0(L_1) < \lambda_0(L_2)$.

On the other hand, it is an open question what happens for graphs that are subexponentially growing, i.e., $\mu = 0$, but not uniformly subexponentially growing.

6.3. Consequences of uniform subexponential growth.

Lemma 6.2. *Assume the graph has uniform subexponential growth. Then, for all $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that*

(a) $m(x) \le Ce^{\varepsilon d(x,y)} m(y)$, *for all* $x, y \in X$,
(b) $\# B_r(x) \le Ce^{\varepsilon r}$, *for all* $r \ge 0$, *where* $\# B_r(x)$ *denotes the number of vertices in* $B_r(x)$.
(c) $\sum_{y \in X} e^{-\varepsilon d(x,y)} \le C$, *for all* $x \in X$.

Proof. To prove (a) let $x, y \in X$. Using the uniform subexponential growth assumption and $x \in B_{d(x,y)}(y)$ yields $m(x) \le m(B_{d(x,y)}(y)) \le Ce^{\varepsilon d(x,y)} m(y)$. Turning to (b), let $x \in X$ and $r \ge 0$. We obtain using (a) and the uniform subexponential growth assumption

$$\# B_r(x) = \sum_{y \in B_r(x)} m(y)/m(y) \le Ce^{\varepsilon r} \frac{m(B_r(x))}{m(x)} \le C^2 e^{2\varepsilon r}.$$

The final statement (c) follows also by direct calculation using (b) (with ε_1)

$$\sum_{y \in X} e^{-\varepsilon d(x,y)} = \sum_{r=1}^{\infty} \sum_{y \in B_r(x) \setminus B_{r-1}(x)} e^{-\varepsilon d(x,y)} \le \sum_{r=1}^{\infty} e^{-\varepsilon(r-1)} \# B_r(x) \le C \sum_{r=1}^{\infty} e^{(\varepsilon_1 - \varepsilon) r}$$

Hence, choosing $\varepsilon_1 = \varepsilon/2$ yields the statement. □

Remark. (a) Lemma 6.2 (b) implies finiteness of distance balls. On the other hand, finite jump size s implies that for each vertex x all neighbors of x are contained in $B_s(x)$. Hence, graphs with uniform subexponential growth and finite jump size are locally finite.

(b) Finiteness of distance balls has strong consequences on the uniqueness of selfadjoint extensions. In particular, by [24, Corollary 1] implies that Q is the maximal form on ℓ^2 and that the restriction of L_2 to $C_c(X)$ (whenever $C_c(X) \subseteq D(L_2)$) is essentially selfadjoint.

In the following we discuss examples to clarify the relation between uniform subexponential growth and bounded geometry in the discrete setting.

Recall that we speak of bounded geometry if n/m is a bounded function which is a natural adaption to the situation of weighted graphs. In Examples 6.3 and 6.4 below, we show that there are graphs with uniform subexponential growth and unbounded geometry. For completeness we also give a example of bounded geometry and exponential growth which is certainly well-known.

Example 6.3 (Uniform subexponential growth and unbounded geometry)**.** Let $X = \mathbb{N}$, $m \equiv 1$, $c \equiv 0$ and consider b such that $b(x, y) = 0$ for $|x - y| \ne 1$, $b(x, x + 1) = x$ for $x \in 4\mathbb{N}$ and $b(x, x + 1) = 1$ otherwise. Clearly, $(n/m)(x) = n(x) = x + 1$ for $x \in 4\mathbb{N}$ and, thus, the graph has unbounded geometry. In particular, L_p is unbounded for all

$p \in [1, \infty]$. Moreover, let ρ be the path metric induced by the edge weights $w(x, x+1) = (n(x) \vee n(x+1))^{-\frac{1}{2}}$. Then, ρ is intrinsic and we obtain that $\rho(x, y) \geq (|x-y| - 3)/(4\sqrt{2})$ for all $x, y \in X$. Hence, $m(B_r(x)) = \#B_r(x) \leq \sqrt{2}(8r + 6)$ which implies uniform subexponential growth.

Example 6.4 (Exponential growth and bounded geometry)**.** Take a regular tree, $c \equiv 0$, set b to be one on the edges and zero otherwise and let $m \equiv 1$. This graph has bounded geometry but is clearly of exponential growth.

It is apparent that the graph in Example 6.3 (a) above has bounded combinatorial vertex degree and the unbounded geometry is induced by the edge weights. So, one might wonder whether there are also examples of uniform subexponential growth but with unbounded combinatorial vertex degree which is the criterion for unbounded geometry in the classical setting. The proposition below shows that this is impossible under the assumptions of uniform subexponential growth and finite jump size. Recall that by the remark below Lemma 6.2 we already know that the graph must be locally finite.

The *combinatorial vertex degree* deg is the function that assigns to each vertex the number of neighbors, that is $\deg(x) = \#\{y \in X \mid b(x, y) > 0\}$, $x \in X$.

Proposition 6.5. *If the graph has uniform subexponential growth with respect to a metric with finite jump size s, then the combinatorial vertex degree is bounded.*

Proof. Suppose the graph has unbounded vertex degree, i.e., there is a sequence of vertices (x_n) such that $\deg(x_n) \geq n^2$ for all $n \geq 1$. We show that there is a sequence of vertices z_n such that $m(B_s(z_n))/m(z_n)$ is unbounded and thus the graph does not have uniform subexponential growth.

If, for $n \geq 1$, there is a neighbor y_n of x_n such that $m(y_n) \leq m(x_n)/\sqrt{\deg(x_n)}$, then we estimate using $x_n \in B_s(y_n)$

$$\frac{m(B_s(y_n))}{m(y_n)} \geq \frac{m(x_n)}{m(y_n)} \geq \sqrt{\deg(x_n)} \geq n$$

We set $z_n = y_n$ in this case. If, on the other hand, $m(y) \geq m(x_n)/\sqrt{\deg(x_n)}$ for all neighbors y of x_n, then

$$\frac{m(B_s(x_n))}{m(x_n)} \geq \frac{1}{m(x_n)} \deg(x_n) \frac{m(x_n)}{\sqrt{\deg(x_n)}} \geq \sqrt{\deg(x_n)} \geq n$$

and set $z_n = x_n$ in this case. Hence, we have proven the claim. $\square$

Corollary 6.6. *Assume there is $D > 0$ such that $b \leq D$ and $m \geq 1/D$. Then, uniform subexponential growth with respect to a metric with finite jump size implies bounded geometry.*

Proof. One simply observes that $n/m \leq D^2 \deg$ and the statement follows from the proposition above. □

The lemma and the corollary above mean for the standard Laplacians $\Delta_p\varphi(x) = \sum_{y\sim x}(\varphi(x)-\varphi(y))$ on $\ell^p(X,1)$ and $\Delta_p^{(n)}\varphi(x) = \frac{1}{\deg(x)}\sum_{y\sim x}(\varphi(x)-\varphi(y))$ on $\ell^p(X,\deg)$, that uniform subexponential growth implies bounded geometry, both in the sense of bounded n/m and also in the sense of bounded deg.

6.4. **Lipschitz continuous functions.** We denote by C_{Lip} the real valued bounded Lipschitz continuous functions with Lipschitz constant $\varepsilon > 0$ with respect to an intrinsic metric ρ, i.e.,

$$C_{\mathrm{Lip}} := \{\psi : X \to \mathbb{R} \mid \psi(x) - \psi(y) \leq \varepsilon\rho(x,y),\ x, y \in X\} \cap \ell^\infty(X,m).$$

Lemma 6.7. *Let (b,c) be a connected graph over (X,m) and ρ be an intrinsic metric which has finite jump size (J). Let $\varepsilon > 0$ and let s be the jump size of ρ. Then, for all $\psi \in C_{\mathrm{Lip}}$,*

(a) *e^ψ is a bounded Lipschitz continuous function, in particular, $e^\psi D(Q) = D(Q)$.*
(b) *$|1 - e^{\psi(x)-\psi(y)}| \leq \varepsilon e^{\varepsilon s}\rho(x,y)$, for $x \sim y$.*
(c) *$|(e^{-\psi(x)} - e^{-\psi(y)})(e^{\psi(x)} - e^{\psi(y)})| \leq 2\varepsilon^2 e^{\varepsilon s}\rho(x,y)^2$, for $x \sim y$.*

Proof. The first statement of (a) follows from mean value theorem, that is for any $x, y \in X$ we have

$$|e^{\psi(x)} - e^{\psi(y)}| \leq |\psi(x) - \psi(y)|e^{\|\psi\|_\infty} \leq \varepsilon\rho(x,y)e^{\|\psi\|_\infty}.$$

Now, $e^\psi D(Q) \subseteq D(Q)$ is a consequence of [13, Lemma 3.5]. The other inclusion follows since $e^{-\psi}$ is also bounded and Lipschitz continuous. Similarly, we get (b) using the Taylor expansion of the exponential function

$$|1 - e^{\psi(x)-\psi(y)}| = \sum_{k\geq 1}\frac{(\psi(x)-\psi(y))^k}{k!} \leq \varepsilon\rho(x,y)\sum_{k\geq 1}\frac{(\varepsilon s)^{k-1}}{k!} \leq \varepsilon\rho(x,y)e^{\varepsilon s},$$

and, similarly, using $|(e^{-\psi(x)} - e^{-\psi(y)})(e^{\psi(x)} - e^{\psi(y)})| = 2\sum_{k\in 2\mathbb{N}}\frac{(\psi(x)-\psi(y))^k}{k!}$ we get (c). □

6.5. **Kernels.** Let $A : D(A) \subseteq \ell^p \to \ell^q$, $p, q \in [1,\infty]$ be a densely defined linear operator. We denote by $\|A\|_{p,q}$ the operator norm of A, i.e.

$$\|A\|_{p,q} = \sup_{f\in D(A),\ \|f\|_p=1}\|Af\|_q.$$

Note that any such operator $A : D(A) \subseteq \ell^p \to \ell^q$, $p < \infty$, with $C_c(X) \subseteq D(A)$ admits a kernel $k_A : X \times X \to \mathbb{C}$ such that

$$Af(x) = \sum_{x\in X} k_A(x,y)f(y)m(y)$$

for all $f \in D(A)$, $x \in X$, which can be obtained by

$$k_A(x,y) = \frac{1}{m(x)m(y)}\langle A1_y, 1_x\rangle,$$

where $1_v(w) = 1$ if $w = v$ and $1_v(w) = 0$ otherwise.

We recall the following well known lemma which shows that the operator norm of $A : \ell^p \to \ell^q$ can be estimated by its integral kernel.

Lemma 6.8. *Let $p, p^* \in [1,\infty)$, $p^{-1} + p^{*-1} = 1$, and let A be a densely defined linear operator with $C_c(X) \subseteq D(A) \subseteq \ell^p$. Then,*

(a) $\|A\|_{p,q} \le \left(\sum_y \|k_A(\cdot,y)\|_q^{p^*} m(y)\right)^{\frac{1}{p^*}}$ *for $q < \infty$ and $p \in (1,\infty)$, and*
$\|A\|_{1,q} \le \sup_y \|k_A(\cdot,y)\|_q$ *for $q < \infty$.*

(b) $\|A\|_{p,\infty} \le \sup_x \|k_A(x,\cdot)\|_{p^*}$ *and equality holds if $p = 1$.*

Proof. Statement (a) follows from the fact $\|f\|_p = \sup_{g\in\ell^{p^*},\,\|g\|_{p^*}=1}\langle g, f\rangle$ and twofold application of Hölder inequality. The first part of (b) follows simply from Hölder inequality. For the second part note that $\|A\|_{1,\infty} \ge \sup_{x,y\in X} |A1_y(x)|/m(y)$. □

6.6. **Heat kernel estimates.** We denote the kernel of the semigroup T_t, $t \ge 0$ by p_t. As the semigroups are consistent on ℓ^p, $p \in [1,\infty]$, i.e., they agree on their common domains, the kernel p_t does not depend on p.

The following heat kernel estimate will be the key to p-independence of spectra of L_p. It is proven in [4], for locally finite graphs and $c \equiv 0$. However, on the one hand, local finiteness is not used in [4] for this result and, on the other hand, the remark below Lemma 6.2 shows that we are in the local finite situation anyway whenever we assume uniform subexponential growth. We conclude the statement for $c \ge 0$ by a Feynman-Kac formula.

Lemma 6.9. *Let (b,c) be a connected graph over (X,m) and ρ be an intrinsic metric such that the balls are finite (B) and which has finite jump size (J). We have for all $t \ge 0$ and $x, y \in X$*

$$p_t(x,y) \le (m(x)m(y))^{-\frac{1}{2}} e^{-\rho(x,y)\log\frac{\rho(x,y)}{2et}}.$$

Proof. Denote the semigroup of the graph $(b,0)$ by $T_t^{(0)}$ and the kernel by $p_t^{(0)}$ and correspondingly for (b,c) by T_t and p_t.

For $c \equiv 0$ the estimate above is found in [4, Theorem 2.1] for $p_t^{(0)}$. Now, by a Feynman-Kac formula, see e.g. [14], we have

$$p_t(x,y) = T_t\delta_y(x) = \mathbb{E}_x\big[e^{-\int_0^t \frac{c}{m}(\mathbb{X}_s)ds}\delta_y(\mathbb{X}_t)\big] \le \mathbb{E}_x\big[\delta_y(\mathbb{X}_t)\big] = T_t^{(0)}\delta_y(x) = p_t^{(0)}(x,y),$$

where $\delta_y = 1_y/m(y)$. This proves the claim. □

By basic calculus, we obtain the following heat kernel estimate.

Lemma 6.10. *Let (b,c) be a connected graph over (X,m) and ρ be an intrinsic metric such that (B) and (J). For all $\beta>0$ there exists a constant $C(\beta)$ such that for all $t\geq 0$, $x,y\in X$*

$$p_t(x,y)\leq (m(x)m(y))^{-\frac{1}{2}}e^{-\beta\rho(x,y)+C(\beta)t}.$$

Proof. Let $\beta>0$ and $r>0$ let $f(r)=-r\log(r/2e)+\beta r$. Direct calculation shows that the function f assumes its maximum on the domain $(0,\infty)$ at the point $r_0=2e^{\beta}$. In particular, setting $C(\beta)=2e^{\beta}$ yields

$$-\frac{\rho(x,y)}{t}\log\frac{\rho(x,y)}{2et}\leq -\beta\frac{\rho(x,y)}{t}+C(\beta).$$

for all $t>0$ and $x,y\in X$. The statement follows now from the lemma above. □

6.7. **Proof of the theorem.** In this section we prove Theorem 6.1 following the strategy of [42]. The proof is divided into several lemmas and as always we assume that d is an intrinsic metric with finite jump size s.

Lemma 6.11. *Let (b,c) be a connected graph over (X,m) and ρ be an intrinsic metric such (B) and (J). For every compact set $K\subseteq\rho(L_2)$ there is $\varepsilon>0$ and $C<\infty$ such that for all $z\in K$ and all $\psi\in C_{\mathrm{Lip}}$*

$$\|e^{-\psi}(L_2-z)^{-1}e^{\psi}\|_{2,2}\leq C.$$

Proof. Let $\varepsilon>0$ and $\psi\in C_{\mathrm{Lip}}$. By Lemma 6.7 we have $e^{-\psi}D(Q)=e^{\psi}D(Q)=D(Q)$. Let Q_ψ be the (not necessarily symmetric) form with domain $D(Q_\psi)=D(Q)$ acting as

$$Q_\psi(f,g):=Q(e^{-\psi}f,e^{\psi}g)-Q(f,g).$$

Application of Leibniz rule yields, for $f\in D(Q_\psi)$,

$$\begin{aligned}|Q_\psi(f,f)|=&\frac{1}{2}\sum_{x,y\in X}b(x,y)|f(y)|^2(e^{-\psi(x)}-e^{-\psi(y)})(e^{\psi(x)}-e^{\psi(y)})\\&+\frac{1}{2}\sum_{x,y\in X}b(x,y)\overline{f(y)}(f(x)-f(y))(1-e^{\psi(y)-\psi(x)})\\&+\frac{1}{2}\sum_{x,y\in X}b(x,y)f(y)(1-e^{\psi(x)-\psi(y)})\overline{(f(x)-f(y))}.\end{aligned}$$

Applying Cauchy-Schwarz inequality, Lemma 6.7 (b) and (c) and the intrinsic metric property, gives

$$\begin{aligned}\ldots\leq&\varepsilon^2C\sum_{x,y\in X}|f(x)|^2b(x,y)\rho(x,y)^2+2C\varepsilon\Big(\sum_{x,y\in X}|f(x)|^2b(x,y)\rho(x,y)^2\Big)^{\frac{1}{2}}Q(f)^{\frac{1}{2}}\\&\leq C\varepsilon^2\|f\|_2^2+2C\varepsilon\|f\|_2Q(f)^{\frac{1}{2}}.\end{aligned}$$

Hence, the basic inequality $2ab \leq (1/\delta)a^2 + \delta b^2$ for $\delta > 0$ and $a, b \geq 0$ (applied with $a = C\varepsilon\|f\|_2^2$ and $b = Q(f)^{\frac{1}{2}}$) yields

$$|Q_\psi(f,f)| \leq C\varepsilon^2(1 + \tfrac{1}{\delta})\|f\|_2^2 + \delta Q(f).$$

This shows that Q_ψ is Q bounded with bound 0. According to [26, Theorem VI.3.9] this implies that the form $Q_\psi + Q$ is closed and sectorial. It can be checked directly that the corresponding operator is $e^\psi L_2 e^{-\psi}$ with domain $D_\psi = e^\psi D(L_2)$. Moreover, for $K \subseteq \rho(L_2)$ compact, we can choose $\varepsilon, \delta > 0$ that $2\big\|\big(C(1+1/\delta)\varepsilon^2 e^{2s} + \delta L_2\big)\cdot(L_2 - z)^{-1}\big\|_{2,2} < 1$ for all $z \in K$ since C is a universal constant. Therefore, again by [26, Theorem VI.3.9] this implies existence of $C = C(K, \varepsilon)$ such that

$$\|e^{-\psi}(L_2 - z)^{-1}e^\psi\|_{2,2} = \|(e^\psi L_2 e^{-\psi} - z)^{-1}\|_{2,2} \leq C$$

for all $z \in K$ and $\psi \in C_{\mathrm{Lip}}$. □

Let us recall some well known facts about consistency of semigroups and resolvents. The semigroups T_t are consistent on ℓ^p. By the spectral theorem the Laplace transform for the resolvent $G_z = (L_2 - z)^{-1}$

$$G_z f = \int_0^\infty e^{zt} T_t f \, dt,$$

holds for f in ℓ^2 and $z \in \{w \in \mathbb{C} \mid \Re w < 0\}$ (the open left half plane). By density and duality arguments, this formula extends to f in ℓ^p, $p \in [1, \infty)$, in the strong sense and to $p = \infty$ in the weak sense. This shows that the resolvents $(L_p - z)^{-1}$ are consistent on ℓ^p for $z \in \{w \in \mathbb{C} \mid \Re w < 0\}$.

We denote by g_α the kernel of the resolvent $G_\alpha = (L_p - \alpha)^{-1}$ which is independent of $p \in [1, \infty]$ for $\alpha < 0$.

Lemma 6.12. *Let (b, c) be a connected graph over (X, m) and ρ be an intrinsic metric such that (B), (J) and assume the graph has uniform subexponential volume growth. For any $\varepsilon > 0$ there exists $\alpha < 0$ and $C < \infty$ such that*

(a) $|g_\alpha(x,y)| \leq C(m(x)m(y))^{-\frac{1}{2}} e^{-\varepsilon\rho(x,y)}$ *for all $x, y \in X$,*
(b) $\|e^\psi G_\alpha e^{-\psi} m^{\frac{1}{2}}\|_{1,2} \leq C$ *for all $\psi \in C_{\mathrm{Lip}}$,*
(c) $\|m^{\frac{1}{2}} e^\psi G_\alpha e^{-\psi}\|_{2,\infty} \leq C$ *for all $\psi \in C_{\mathrm{Lip}}$.*

Proof. (a) By the Laplace transform of the resolvent and Lemma 6.10, we get

$$g_\alpha(x,y) = \int_0^\infty e^{\alpha t} p_t(x,y) dt \leq (m(x)m(y))^{-\frac{1}{2}} e^{-\varepsilon\rho(x,y)} \int_0^\infty e^{(\alpha + C)t} dt,$$

which yields the statement for $\alpha < -C$.
(b) By Lemma 6.8 (a), $\psi \in C_{\mathrm{Lip}}$, part (a) above (with $\varepsilon_1 \geq 2\varepsilon$) and

Lemma 6.2 (c)

$$\|e^{\psi}G_\alpha e^{-\psi}m^{\frac{1}{2}}\|_{1,2}^2 \leq \sup_{y\in X}\|g_\alpha(\cdot,y)e^{\psi(\cdot)-\psi(y)}m(y)^{\frac{1}{2}}\|_2^2$$
$$\leq C\sup_{y\in X}\sum_{x\in X}e^{(2\varepsilon-2\varepsilon_1)\rho(x,y)} < \infty.$$

The proof of (c) works similarly using Lemma 6.8 (b). □

Lemma 6.13. *Let (b,c) be a connected graph over (X,m) and ρ be an intrinsic metric such that (B), (J) and assume the graph has uniform subexponential volume growth. Then, $(L_2-z)^{-2}$ extends to a bounded operator on ℓ^p for all $z\in\rho(L_2)$ and $p\in[1,\infty]$. Moreover, for all compact $K\subseteq\rho(L_2)$ there is $C<\infty$ such that for all $z\in K$ and $p\in[1,\infty]$*

$$\|(L_2-z)^{-2}\|_{p,p}\leq C.$$

Proof. For $z\in\rho(L_2)$ denote by $g_z^{(2)}$ the kernel of the squared resolvent $(G_z)^2=(L_2-z)^{-2}$. Applying the resolvent identity twice yields

$$(G_z)^2=(G_\alpha+(z-\alpha)G_\alpha G_z)(G_\alpha+(z-\alpha)G_zG_\alpha)=G_\alpha(I+(z-\alpha)G_z)^2G_\alpha,$$

for all $\alpha<0$. Therefore,

$$m^{\frac{1}{2}}e^{\psi}(G_z)^2e^{-\psi}m^{\frac{1}{2}}=\Big(m^{\frac{1}{2}}e^{\psi}G_\alpha e^{-\psi}\Big)\Big(I+(z-\alpha)e^{\psi/2}G_ze^{-\psi/2}\Big)^2\Big(e^{\psi}G_\alpha e^{-\psi}m^{\frac{1}{2}}\Big),$$

for all $\varepsilon>0$ and $\psi\in C_{\mathrm{Lip}}$. Taking the norm $\|\cdot\|_{1,\infty}$ and factorizing $\|\ldots\|_{1,\infty}\leq\|(\ldots)\|_{2,\infty}\|(\ldots)\|_{2,2}\|(\ldots)\|_{1,2}$ yields that $U:=m^{\frac{1}{2}}e^{\psi}G_z^2e^{-\psi}m^{\frac{1}{2}}$ is a bounded operator $\ell^1\to\ell^\infty$ by Lemma 6.11 and Lemma 6.12 with appropriate choice of $\alpha<0$, $\varepsilon>0$ and all $\psi\in C_{\mathrm{Lip}}$. Hence, the operator U admits a kernel $k_U(x,y)=(m(x)m(y))^{\frac{1}{2}}e^{\psi(x)-\psi(y)}g_z^{(2)}(x,y)$, $x,y\in X$, and we conclude from Lemma 6.8 (b) that

$$|g_z^{(2)}(x,y)|\leq C(m(x)m(y))^{-\frac{1}{2}}e^{\psi(y)-\psi(x)}.$$

For chosen $\varepsilon>0$ and any fixed $x,y\in X$ let $\psi:u\mapsto\varepsilon(\rho(u,y)\wedge\rho(x,y))$ and we obtain from Lemma 6.2 (a) (with ε)

$$|g_z^{(2)}(x,y)|\leq C(m(x)m(y))^{-\frac{1}{2}}e^{-\varepsilon\rho(x,y)}\leq Cm(x)^{-1}e^{-\frac{\varepsilon}{2}\rho(x,y)}.$$

Thus, using Lemma 6.8 (a) and Lemma 6.2 (c), we obtain

$$\|G_z^2\|_{1,1}\leq\sup_{y\in X}\sum_{x\in X}|g_z^{(2)}(x,y)|m(x)\leq C\sup_{y\in X}\sum_{x\in X}e^{-\frac{\varepsilon}{2}\rho(x,y)}<\infty.$$

As G_z^2 is bounded for $p=1$ and $p=2$, it follows from the Riesz-Thorin interpolation theorem that it is bounded for $p\in[1,2]$ and by duality for $p\in[1,\infty]$. □

Lemma 6.14. *Let (b,c) be a connected graph over (X,m). If $\sigma(L_p)\subseteq[0,\infty)$ for all $p\in[1,\infty]$, then $\sigma(L_2)\subseteq\sigma(L_p)$.*

Proof. The operators $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are consistent for $z \in \{w \in \mathbb{C} \mid \Re w < 0\} \subseteq \rho(L_p) = \rho(L_{p^*})$ by the discussion above Lemma 6.12 for all $p \in [1,\infty]$. By the assumption $\sigma(L_p) \subseteq [0,\infty)$ the resolvent sets are connected which yields by [17, Corollary 1.4] that $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are consistent for $z \in \rho(L_p) \cap \rho(L_q)$ for $p, q \in [1,\infty]$. Moreover, by the standard theory $(L_p - z)^{-1}$ and $(L_q - z)^{-1}$ are analytic on $\rho(L_p) = \rho(L_{p^*})$. By the Riesz-Thorin theorem these resolvents can be consistently extended to analytic ℓ^2-bounded operators. That is, as a ℓ^2-bounded operator-valued function $(L_p - z)^{-1}$ is analytic on $\rho(L_p)$ which is consistent with $(L_2 - z)^{-1}$ on $\rho(L_p) \cap \rho(L_2)$. Note that, $(L_2 - z)^{-1}$ is analytic on $\rho(L_2)$ which is also the maximal domain of analyticity. Thus, the statement follows by unique continuation. □

Proof of Theorem 6.1. We start by showing $\sigma(L_p) \subseteq \sigma(L_2)$. For the kernel $g_z^{(2)}$ of $(L_2 - z)^{-2}$ and fixed $x, y \in X$ the function $\rho(L_2) \to \mathbb{C}$, $z \mapsto g_z^{(2)}(x,y)$ is analytic. By Lemma 6.13 we know that for any compact $K \subseteq \rho(L_2)$ the operators $(L_2 - z)^{-2}$, $z \in K$ are bounded on ℓ^p, $p \in [1,\infty]$. Therefore $(L_2 - z)^{-2}$ is analytic as a family of ℓ^p-bounded operators for $z \in \rho(L_2)$. On the other hand, $(L_p - z)^{-2}$ is analytic as a family of ℓ^p-bounded operators with domain of analyticity $\rho(L_p)$ by [17, Lemma 3.2]. Since $(L_2 - z)^{-2}$ and $(L_p - z)^{-2}$ agree on $\{w \in \mathbb{C} \mid \Re w < 0\}$, by unique continuation they agree as analytic ℓ^p operator-valued functions on $\rho(L_2)$. As the domain of analyticity of $(L_p - z)^{-2}$ is $\rho(L_p)$, this implies $\rho(L_2) \subseteq \rho(L_p)$.

On the other hand, since $\sigma(L_p) \subseteq \sigma(L_2) \subseteq [0,\infty)$ the statement $\sigma(L_2) \subseteq \sigma(L_p)$ follows from Lemma 6.14. □

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