Matthias Keller

Curvature and spectrum for graphs

Classically discreteness of spectrum for Schrödinger operators can result from the potential or from the geometry. The case of a Schrödinger operator $H = \Delta + v$ on $\mathbb{R}^d$ with a non-negative potential v was considered by Friedrichs in 1934, [Fr]. He showed that if v tends uniformly to ∞ in every direction, then H has purely discrete spectrum. A complete characterization for the phenomena of discreteness of the spectrum due to a unbounded potential was given by Mazya/Shubin in 2005 [MS]. See also Lenz/Stollmann/Wingert [LSW] for a corresponding result in the context of Dirichlet forms. On the other hand, Donnelly/Li [DL] showed in 1979 that if the sectional curvatures of a complete, simply connected, negatively curved Riemannian manifold tend uniformly to $-\infty$, then the Laplace Beltrami operator has purely discrete spectrum.

Our first aim is to give a discrete analogue of the Donnelly/Li theorem in the framework of graphs. Secondly, we aim for a unified treatment of decreasing curvature and increasing potential.

Let $G = (V, E)$ be an infinite, simple, connected, locally finite graph. For a finitely supported function $u : V \to \mathbb{R}$, let the quadratic form Q be given by

$$Q(u) = \sum_{\{x,y\}\in E} (u(x) - u(y))^2.$$

There are two natural choices of a measure on V. One is constantly 1 and the other one is the vertex degree function deg. So, firstly the positive selfadjoint operator Δ associated to the completion of Q in $\ell^2(V)$ acts as

$$\Delta u(x) = \sum_{x\sim y}(u(x) - u(y))$$

on $D(\Delta) = \{u \in \ell^2(V) \mid \Delta u \in \ell^2(V)\}$. It is not hard to see that Δ is bounded iff the vertex degree function is bounded. Secondly, taking the completion of Q in $\ell^2(V, \deg)$ yields the bounded positive selfadjoint operator $\widetilde{\Delta}$ acting as

$$\widetilde{\Delta} u(x) = \frac{1}{\deg(x)} \sum_{x\sim y}(u(x) - u(y))$$

which satisfies $0 \le \widetilde{\Delta} \le 2$.

The Cheeger constant α_U for $U \subseteq V$ is defined as

$$\alpha_U := \inf_{W\subseteq U \text{ finite}} \frac{|\partial_E W|}{\mathrm{vol}(W)},$$

where $\partial_E W := \{\{x, y\} \in E \mid x \in W, y \in V \setminus W\}$ and $\mathrm{vol}(W) = \sum_{x\in W} \deg(x)$. Clearly, $0 \le \alpha_U < 1$ and $\alpha_U \le \alpha_{U'}$ for $U' \subseteq U \subseteq V$. We define following [Fu]

$$\alpha_\infty := \lim_{K\subset V \text{ finite}} \alpha_{V\setminus K},$$

where the limit is taking along the net of finite subsets of V. In [Fu], Fujiwara proved the following remarkable theorem:

Theorem 1. *(Fujiwara '96) The essential spectrum of $\widetilde{\Delta}$ is $\{1\}$ iff $\alpha_\infty = 1$.*

The key ingredient of the proof are the Cheeger estimates $1-\sqrt{1-\alpha_U^2} \leq \widetilde{\Delta}_U \leq \min\{1+\sqrt{1-\alpha_U^2}, \alpha_U\}$, $U \subseteq V$ due to Mohar, Dodziuk/Karp and Dodziuk/Kendall. This estimate implies that $\alpha_\infty = 1$ is equivalent to compactness of the operator $P = \widetilde{\Delta} - I$.

Since $\widetilde{\Delta}$ is a bounded infinite dimensional operator it cannot have purely discrete spectrum. Thus, in order to get a better analogy to the theorem of Donnelly/Li, one has to consider the operator Δ in the case of unbounded vertex degree. In [K1] such a theorem is proven. To this end, let

$$d_\infty := \lim_{K\subset V \text{ finite}} \min_{x\in V\setminus K} \deg(x).$$

Theorem 2. *(K. '10) Let $\alpha_\infty > 0$. Then, the essential spectrum of Δ is empty iff $d_\infty = \infty$.*

The proof uses an estimate on the bottom of the spectrum of Δ by the minimum of the vertex degree times the bottom of the spectrum of $\widetilde{\Delta}$. Since such an estimate holds on all complements of finite sets one gets an estimate for the bottoms of the essential spectra.

The theorem gives an analogue to the Donnelly/Li theorem as $\alpha_\infty > 0$ can be understood as a negative curvature assumption and $d_\infty = \infty$ as the curvature tending to $-\infty$. In the framework of planar tessellations the analogue is even clearer.

Assume G is a planar tessellation which is embedded into a topological surface $S \cong \mathbb{R}^2$. Let the set of faces F be given by the closures of the connected components of $S \setminus \bigcup E$. The face degree $\deg(f)$, $f \in F$, is defined as the number of vertices contained in f. The curvature $\kappa : V \to \mathbb{R}$ is given by

$$\kappa(x) := 1 - \frac{\deg(x)}{2} + \sum_{f\in F,\, x\in f} \frac{1}{\deg(f)},$$

which can be understood as an angle defect. Let

$$\kappa_\infty := \lim_{K\subset V \text{ finite}} \sup_{x\in V\setminus K} \kappa(x).$$

Theorem 3. *(K. '10) Let G be a planar tessellation. Then, the essential spectrum of Δ is empty iff $\kappa_\infty = -\infty$. Moreover, $\kappa_\infty = -\infty$ implies that the essential spectrum of $\widetilde{\Delta}$ is equal to $\{1\}$.*

The proof given in [K1] uses that $\kappa_\infty = -\infty$ iff $d_\infty = \infty$ and a Cheeger estimate of the form $\alpha_\infty \geq 1 - 6/\inf_x \deg(x)$. More subtle estimates using curvature can be found in [K1, K2, KP].

Finally, to consider Schrödinger operators $H = \Delta + v$ with $v : V \to [0,\infty)$, let

$$\gamma_\infty := \lim_{K\subset V \text{ finite}} \inf_{W\subseteq V\setminus K \text{finite}} \frac{|\partial_E W| + v(W)}{\operatorname{vol}(W)},$$

where $v(W) = \sum_{x\in W} v(x)$,

$$g_\infty := \lim_{K\subset V \text{ finite}} \inf_{x\in V\setminus K} (\deg + v)(x).$$

and if G is a planar tessellation

$$k_\infty := \lim_{K\subset V \text{ finite}} \inf_{x\in V\setminus K} (-\kappa + v)(x).$$

Our second aim was to give a unified treatment of the relation of uniform growth of curvature and the potential and discreteness of H. From [KL] we can deduce

Theorem 4. *(K./Lenz '10) (1.) Let $\gamma_\infty > 0$. Then, the essential spectrum of H is empty iff $g_\infty = \infty$.*
(2.) Let G be a planar tessellation. Then, the essential spectrum of H is empty iff $k_\infty = \infty$.

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