Notes
Applications of operator theory: Discrete operators

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July 16, 2012

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Chapter 1

Introduction

In this chapter we discuss where we are coming from and where we are heading to. We consciously avoid rigorous terms at first to put through the big picture. All notions will be introduced rigorously in the chapters that follow.

1.1 Applied operator theory - The big picture

While the clarity of linear algebra stems from its restriction to studying linear mappings on finite dimensional vector spaces, the beauty of analysis is expressed as saying that it is the 'art of taking limits'. Operator theory combines this clarity and beauty as it is concerned with the study of linear maps on infinite dimensional space

The fundamental observations for the development is that differentiation (and integration) are linear. The idea is to extend ideas from linear algebra to study functional equations such as

- Schrödinger equation (stationary and time dependent)
- Heat equation

which are real world problems coming from physics. Let us discuss these equations shortly. Let $\Omega \subseteq \mathbb{R}^d$ be open, $\Delta = -\sum_{i=1}^d \partial_{x_i}^2$ the Laplacian and $V : \Omega \to [0, \infty)$. (More generally we let Ω be a Riemannian manifold and Δ the Laplace-Beltrami operator.)

The stationary Schrödinger equation. Find $E \in \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ smooth with $\int_\Omega |f|^2 dx < \infty$ such that

$$\Delta f(x) + V(x)f(x) = Ef(x), \quad x \in \Omega.$$

This equation models the state of a quantum mechanical particle (e.g. an electron) in a certain media. In particular,

Δ ... kinetic energy,

V ... potential energy,

f ... wave function, in particular the probability that the electron in the state f can be found in $A \subseteq \Omega$ is given by $\int_A |f|^2 dx$,

$E \in \mathbb{R}$ ... total energy. The set of all E for which there is a 'suitable' solution f is the set of all energies the system can assume.

The map

$$H : f \mapsto \Delta f + Vf$$

is called a Hamiltonian or Schrödinger operator. It is obviously linear, so the Schrödinger equation

$$Hf = Ef$$

can be viewed as an 'eigenvalue equation'.

The time dependent Schrödinger equation. For given f_0 find $f : \Omega \times \mathbb{R} \to \mathbb{R}$ smooth with $\int_\Omega |f(x,t)|^2 dx < \infty$ for all $t \in \mathbb{R}$ such that

$$Hf = i\partial_t f, \qquad f(\cdot, 0) = f_0.$$

This equation models the time evolution of a quantum mechanical particle which started at time zero in in state f_0. The equation can be formally solved by f given by

$$f(x,t) = e^{itH} f_0(x).$$

The heat equation. For given f_0 find f as above

$$Hf = -\partial_t f, \qquad f(\cdot, 0) = f_0.$$

This equation models the distribution of heat f in Ω which started as f_0 in dependence of time. The equation can be formally solved by f given by

$$f(x,t) = e^{-tH} f_0(x).$$

If H was a symmetric matrix on a $(n+1)$ dimensional vector space, we could diagonalize it in order to solve the problems above, i.e., if $\lambda_0 \leq \ldots \leq \lambda_n$ are the eigenvalues and $U = (u_0, \ldots, u_n)$ is the matrix with the eigenvectors, then

$$H = U \begin{pmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^*$$

This would immediately solve the stationary Schrödinger equation. Moreover for a function $\varphi : \mathbb{R} \to \mathbb{R}$ we can define

$$\varphi(H) = U \begin{pmatrix} \varphi(\lambda_0) & & \\ & \ddots & \\ & & \varphi(\lambda_n) \end{pmatrix} U^*$$

which solves the time dependent Schrödinger equation and the heat equation. Of particular importance is the smallest eigenvalue λ_0:

- For the Schrödinger equations it models the ground state energy which is the lowest energy the system can assume.
- For the heat equation λ_0 determines the long term behavior, i.e. $e^{-tH} = e^{-\lambda_0} I + \text{lower order terms}$.

There is the following news:

- These operators are linear and function spaces are vector spaces ☺
- 'Diagonalization' is not possible for any operator on any function space ☹
- The spectral theorem says that selfadjoint operators on Hilbert spaces can be diagonalized. The 'eigenvalues' are called the spectrum. ☺
- The spectrum is usually impossible to calculate ☹
- We are mathematicians - we can make assumptions and reformulate the questions ☺

1.2 Spectrum of discrete Laplacians - The plan

We are concerned with the spectrum of discrete analogues of the Laplacian $\mathcal{L}$. These are difference operators on graphs. The plain vanilla model is as follows: The underlying space X is discrete and its elements are called the vertices. Vertices are connected by edges which are a subset of $X \times X$. For a function $\varphi : X \to \mathbb{R}$ the operator $\mathcal{L}$ acts as

$$\mathcal{L}\varphi(x) = \sum_{y \sim x} (\varphi(x) - \varphi(y)).$$

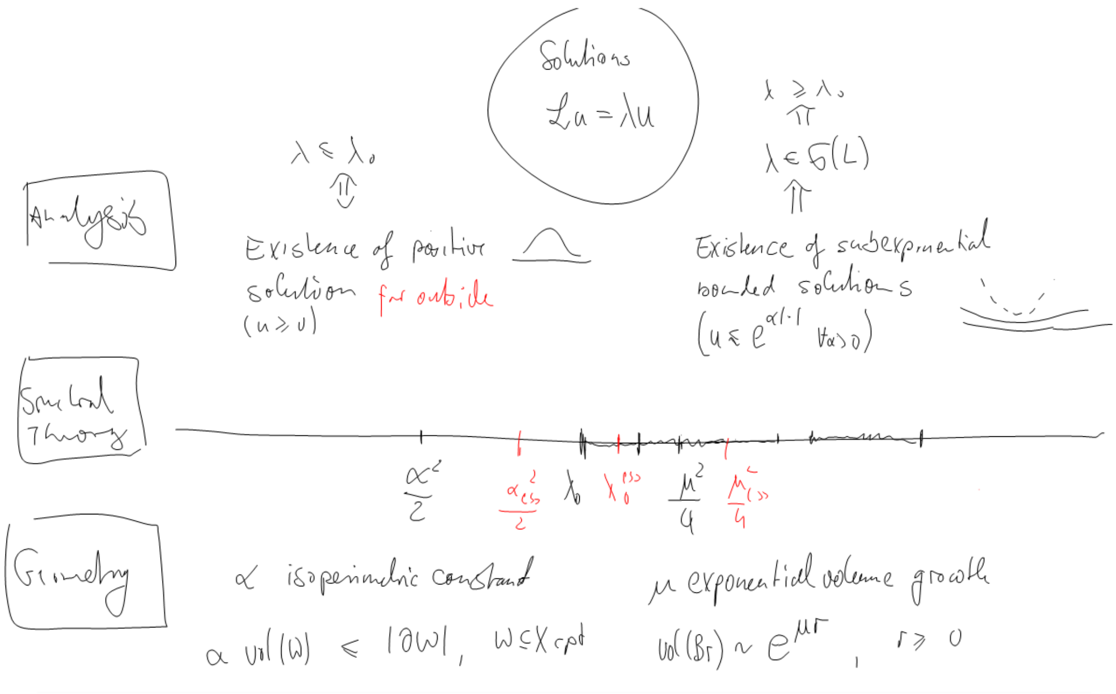
The first step is to define a selfadjoint restriction L of $\mathcal{L}$ and discuss its basic properties. Then we study the spectrum of $\sigma(L)$ and the essential spectrum $\sigma_{\text{ess}}(L)$. The essential spectrum is the part of the spectrum that is stable under 'small' perturbations.

A particular focus will be put on the bottom of the (essential) spectrum

$$\begin{aligned} \lambda_0 &= \inf \sigma(L) \\ \lambda_0^{\text{ess}} &= \inf \sigma_{\text{ess}}(L) \end{aligned}$$

This involves the analysis of solutions on the one hand and a study of the underlying geometry on the other.

Here is the picture:



Ende 1. Vorlesung

Chapter 2

Toolbox A. Preliminaries

2.1 Basic notions in topology

Literature: Boto von Querenburg, 'Mengentheoretische Topologie'.

Let a set X be given. A topology determines which subsets of X are open.
In particular, a *topology* is a subset $\mathcal{O}$ of the power set $2^X = \{Y \subset X\}$ such that

- $\emptyset, X \in \mathcal{O}$,
- If $O_\iota \in \mathcal{O}$, $\iota \in J$, then $\bigcup_{\iota \in J} O_\iota \in \mathcal{O}$, ($J$ is an arbitrary index set),
- If $O_1, \ldots, O_n \in \mathcal{O}$, then $\bigcap_{i=1}^n O_i \in \mathcal{O}$.

The pair $(X, \mathcal{O})$ is called a topological space. The elements of $\mathcal{O}$ are called *open* sets. The complements $X \setminus O$ of open sets $O \in \mathcal{O}$ are called *closed* sets. A set $U \subseteq X$ is called a neighborhood of $x \in X$ if there is $O \in \mathcal{O}$ such that $x \in O \subseteq U$.

Examples 1. $\{\emptyset, X\}$, the trivial topology.
2. 2^X, the discrete topology (suitable for countable X which are sets such that there exist an injective map $X \to \mathbb{N}$).
3. If (X, d) is a metric space, then the set of open sets with respect to d is a topology. (A subset $A \subseteq X$ is called open with respect to d if for every $x \in A$ there is $\varepsilon > 0$ such that $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\} \subseteq A$). In particular, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ are topological spaces with the Euclidian topology.

A set $K \subseteq X$ is called *compact* if for all open coverings there is a finite subcovering, i.e., for $O_\iota \in \mathcal{O}$ for $\iota \in J$ (where J is in an index set) such that $K \subseteq \bigcup_{\iota \in J} O_\iota$ there exist $O_1, \ldots, O_n \in \{O_\iota\}_{\iota \in J}$ such that $K \subseteq \bigcup_{i=1}^n O_i$.

A space X is called *locally compact* if every point has a compact neighborhood.

A space is called *second countable* if there is a countable subset of $\mathcal{O}$ such that every open set can be written as a union over these sets, (i.e., there is $\{U_i\}_{i \in \mathbb{N}} \subseteq \mathcal{O}$ such that for every $O \in \mathcal{O}$ there is a sequence $(i_k) \in \mathbb{N}^\mathbb{N}$ such that $O = \bigcup_{k \in \mathbb{N}} U_{i_k}$.) Such a subset is called a *basis*.

A space X is called *Hausdorff* if for every $x, y \in X$, $x \neq y$, there exist neighborhoods $O_x, O_y \in \mathcal{O}$ of x and y such that $O_x \cap O_y \neq \emptyset$.

Examples 1. Open subsets of $\mathbb{R}^d/\mathbb{C}^d$ with the topology generated by the Euclidean metric is a locally compact, second countable Hausdorff space.

2. A set X with the discrete topology is a locally compact Hausdorff space. Moreover, X is second countable iff X is countable.

3. A compact metric space is a locally compact, second countable Hausdorff space.

Exercise 1: A second countable space X is *σ-compact*, i.e., X is the countable union of compact sets.

Counter-examples 1. Non locally compact spaces:
1.a. The comb $C \subset \mathbb{R}^2$ (picture) with induced topology by $\mathbb{R}^2$, i.e. $\mathcal{O}_C = \{C \cap O \mid O \in \mathcal{O}_{\mathbb{R}^2}\}$.
1.b. Infinite dimensional Hilbert spaces.

2. The discrete topology on an uncountable set is not second countable.

3. Let ρ be a pseudo metric, that is a symmetric map $\rho : X \times X \to [0, \infty)$ that satisfies the triangle inequality. If $\rho(x, y) = 0$ for some $x \neq y$, then topology generated by the open sets with respect to ρ is not Hausdorff.

We now recall the definition of continuous functions.

Lemma 1. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be topological spaces and $f : X \to Y$. Then, the following are equivalent:*

(i) For every $x \in X$ and every open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $f(U) \subseteq V$.

(ii) For every open set $O \subseteq Y$ the set $f^{-1}(O)$ is open in X.

Proof. (i)$\Rightarrow$(ii): Let $O \subseteq Y$ be open. Let $x \in f^{-1}(O)$ be arbitrary. For $f(x)$ let $V \subseteq O$ be an open neighborhood (i.e., take an arbitrary open set O' containing $f(x)$ and let $V = O' \cap O$). By (i) there is an open neighborhood of U_x of x such that $f(U_x) \subseteq V$. In particular $U_x \subseteq f^{-1}(V) \subseteq f^{-1}(O)$. Hence, $f^{-1}(O) = \bigcup_{x \in f^{-1}(U)} U_x$ is open as it is a union of open sets.
(ii)$\Rightarrow$(i): Take $U = f^{-1}(V)$. $\square$

A function is called *continuous* if it satisfies one (and thus all) of the assumptions of the Lemma above.

Examples 1. On a countable set with the discrete topology every function is continuous (because every set is open).

2. If X and Y are metric spaces then (i) coincides with the ε-δ-definition of continuity.

We denote the set of continuous functions from X to $\mathbb{K}$ by $C(X)$. Moreover, the support $\operatorname{supp} f$ of a function $f : X \to \mathbb{R}$ is the closure of the set where the function does not vanish, i.e.,

$$\operatorname{supp} f = \bigcap \{A \subseteq X \mid \text{closed}, \ f|_{X \setminus A} \equiv 0\}.$$

We call a function *compactly supported* if its support is a compact set and denote the set of those functions by $C_c(X)$.

Exercise 2: $C(X)$ and $C_c(X)$ are vector spaces.

2.2 Measure and integration theory

Literature: Heinz Bauer, 'Maß und Integrationstheorie'.

2.2.1 σ-algebras and measures

Let X be a set. A measure is a map that assigns to subsets a non-negative number which can be thought as the volume (weight, energy, ...). In order to guarantee nice properties one has to make restrictions on the set of measurable sets.

A subset $\mathcal{A}$ of the power set 2^X is called a *σ-algebra* if it satisfies the following properties:

- $X \in \mathcal{A}$.
- If $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$.
- If $A_n \in \mathcal{A}$, $n \in \mathbb{N}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

A set $A \subseteq \mathcal{A}$ is called *measurable* if $A \in \mathcal{A}$.

Examples 1. $\{\emptyset, X\}$,
2. 2^X,
3. $\mathcal{A}_0 = \{A \subseteq X \mid A \text{ or } X \setminus A \text{ countable}\}$.
4. If $\mathcal{A}_\iota$ for ι from some index set J are σ-algebras, then $\bigcap_{\iota\in J} \mathcal{A}_\iota$ is a sigma algebra. For a given subset $\mathcal{E} \subseteq 2^X$ we can define the smallest σ-algebra $\mathcal{A}_\mathcal{E}$ containing $\mathcal{E}$ by the intersection over all σ-algebras containing $\mathcal{E}$.
5. If X is a topological space, then the smallest σ-algebra $\mathcal{B}$ that contains all open sets is a σ-algebra and it is called the *Borel σ-algebra*.

We let $\mathbb{K}^d$ with the Euclidean topology be always equipped with its corresponding Borel σ-algebra. We consider $[0, \infty] = [0, \infty) \cup \{\infty\}$ equipped with the σ-algebra $\mathcal{B}_\infty = \mathcal{B} \cup \{B \cup \{\infty\} \mid B \in \mathcal{B}\}$.

Let $\mathcal{A}$ be a σ-algebra. A map $\mu : \mathcal{A} \to [0, \infty]$ is called a *measure* on $(X, \mathcal{A})$ if

- $\mu(\emptyset) = 0$

- If $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, are mutually disjoint, then $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n=1}^{\mathbb{N}} \mu(A_n)$.

A measure is called *finite* if $\mu(X) < \infty$ and a *probability measure* if $\mu(X) = 1$. We say a property holds μ-almost everywhere (μ-almost surely) on X if there exists a measurable set $X_0 \subseteq X$ with $\mu(X_0) = 0$ such that the property holds for all $x \in X \setminus X_0$.

Example 1. $f : X \to \mathbb{R}$ is $f \equiv 0$ almost surely if there is a measurable X_0 with $\mu(X_0) = 0$ such that $f|_{X \setminus X_0} \equiv 0$.

2. We say a function f is defined almost everywhere on X if there is a set X_0 of measure zero such that f is a function on $X \setminus X_0$ (e.g. $f : X \setminus X_0 \to \mathbb{C}$). The triple $(X, \mathcal{A}, \mu)$ is called a *measure space*. We will often suppress the σ-algebra in notation and simply write (X, μ). A measure space $(X, \mathcal{A}, \mu)$ is called σ-finite if there is a sequence (A_n) of measurable sets such that $\bigcup_{n\in\mathbb{N}} A_n = X$ and $\mu(A_n) < \infty$ for $n \in \mathbb{N}$. (It is the countable union of finite measure spaces).

Ende 2. Vorlesung

Let X be a topological space and $\mathcal{B}$ is the Borel σ-algebra. A measure μ on $(X, \mathcal{B})$ is called a *Borel measure* if $\mu(K) < \infty$ for all compact $K \subset X$. A Borel measure μ on $(X, \mathcal{B})$ is called a *Radon measure* if it is *inner regular* and *locally finite*, i.e., $\mu(B) = \sup_{K \subset B \text{ compact}} \mu(K)$ and for every $x \in X$ there is a neighborhood U of x such that $\mu(U) < \infty$. If X is a second countable Hausdorff space, then inner regularity implies local finiteness. (**Exercise 3**)

Examples 1. The Lebesgue measure Leb on $(\mathbb{R}^d, \mathcal{B})$ is a Radon measure. (The Banach-Tarski paradox shows that a three dimensional unit ball can be decomposed into subsets which can be composed into two unit balls. This shows that these subsets cannot be Lebesgue measurable.) Indeed, $(\mathbb{R}^d, \mathcal{B}, \mathrm{Leb})$ is a σ-finite measure space (choose $A_n = B_n(0)$).

2. Assume X is countable. Let $\mathcal{A} = 2^X$ ($=\mathcal{A}_0$ from Example 3 above). Then, all measures on $(X, 2^X)$ are given by maps $m : X \to [0, \infty]$ via

$$m(A) := \sum_{x\in A} m(x) \;\big(= \sup\big\{ \sum_{x\in A_0} m(x) \mid A_0 \subseteq A \text{ finite}\big\}\big), \quad A \subseteq X.$$

(Indeed, given an arbitrary measure μ on $(X, 2^X)$ define $m(x) = \mu(\{x\})$, $x \in X$.)
The measure space (X, m) is σ-finite if and only if $m : X \to [0, \infty)$.
If X is equipped with the discrete topology, i.e., every set in 2^X is open (and, in particular, every singleton set $\{x\}$ is open), the Borel σ-algebra is given by $\mathcal{B} = 2^X$. If not stated otherwise we will always choose the discrete topology and 2^X as the σ-algebra for a countable set X and restrict ourselves to measures $m : X \to (0, \infty)$. We call a pair (X, m) satisfying these assumptions a discrete measure space.

2.2.2 Measurable and integrable functions

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$. A function $f : X_1 \to X_2$ is called *measurable* if for every measurable set $A \subseteq X_2$ the set $f^{-1}(A) \subseteq X_1$ is measurable, (i.e., $A \in \mathcal{A}_2$ implies $f^{-1}(A) \in \mathcal{A}_2$.)

Mostly, we will consider $(X_2, \mathcal{A}_2, \mu_2)$ to be $(\mathbb{C}, \mathcal{B}, \mathrm{Leb})$, $(\mathbb{R}, \mathcal{B}, \mathrm{Leb})$ or $([0,\infty], \mathcal{B}_\infty, \mathrm{Leb})$.

Let $(X, \mathcal{A}, \mu)$, be given. A function $\varphi : X \to \mathbb{C}$ is called simple if there are $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and measurable sets $A_1, \ldots, A_n \subseteq X$ such that $\varphi = \sum_{i=1}^n \alpha_i 1_{A_i}$, (where 1_A is equal to 1 in A and zero otherwise.) We define the integral for a simple function by

$$\int_X \varphi d\mu = \int_X \varphi(x) d\mu(x) = \sum_{i=1}^n \alpha_i \mu(A_i).$$

For measurable function $f : X \to [0, \infty)$, we define the integral $\int_X f d\mu \in [0, \infty]$ by

$$\int_X f d\mu = \int_X f(x) d\mu(x) = \sup\{\int_X \varphi d\mu \mid 0 \le \varphi \le f,\ \varphi \text{ simple}\}.$$

A function $f : X \to \mathbb{C}$ is called integrable if f is measurable and $\int_X |f| d\mu < \infty$. In this case, we define

$$\int_X f d\mu = \int_X (\operatorname{Re} f)_+ d\mu - \int_X (\operatorname{Re} f)_- d\mu + i \int_X (\operatorname{Im} f)_+ d\mu - i \int_X (\operatorname{Im} f)_- d\mu,$$

where $g_\pm$ of a function $g : X \to \mathbb{R}$ is defined as $g_\pm = (\pm g) \vee 0$.

Example. Let (X, m) be a discrete measure space (i.e., X countable set with discrete topology, σ-algebra 2^X and $m : X \to (0, \infty)$). The integral of $f : X \to [0, \infty)$ is given by

$$\int_X f dm = \sum_{x \in X} f(x) m(x) =: \sum_X fm.$$

2.2.3 The circus theorems

Let (X, μ) be a measure space.

Theorem 1. *(Fatou [Bauer, Lemma 15.2]) Let $f_n : X \to [0, \infty]$, $n \in \mathbb{N}$, measurable. Then,*

$$\int_X \liminf_{n \to \infty} f_n d\mu \le \liminf_{n \to \infty} \int_X f_n d\mu.$$

Counter-examples. Let $f_n = n \cdot 1_{[0, \frac{1}{n}]}$ on $[0, 1]$ and $g_n = \frac{1}{n} \cdot 1_{[0,n]}$ on $[0, \infty)$, $n \in \mathbb{N}$, then $f_n \to 0$ pointwise and $g_n \to 0$ even uniformly. But,

$$\int_{[0,1]} f_n d\mathrm{Leb} = \int_{[0,\infty]} g_n d\mathrm{Leb} = 1$$

Theorem 2. *(Beppo Levi - Monotone convergence[Bauer, Satz 11.4]) Let $f_n : X \to [0,\infty]$, $n \in \mathbb{N}$, measurable and non-decreasing (i.e., $f_n \leq f_{n+1}$ almost everywhere). Then,*

$$\int_X \sup_{n\in\mathbb{N}} f_n d\mu = \sup_{n\in\mathbb{N}} \int_X f_n d\mu.$$

In particular for $f := \limsup_{n\to\infty} f_n$ we have $\int f d\mu = \limsup \int f_n d\mu$.

Theorem 3. *(Lebesgue - Dominated convergence [Bauer, Satz 15.6]) Let $f, f_n : X \to \mathbb{C}$, $n \in \mathbb{N}$, integrable and $f_n \to f$ almost everywhere and $g : X \to [0,\infty]$ integrable such that $|f_n| \leq g$. Then,*

$$\int_X f d\mu = \lim \int_X f_n d\mu.$$

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces. Then $(X, \mathcal{A}, \mu) = (X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$ is defined by

- $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ is the smallest σ-algebra such that the projections $p_i : X \to X_i$, $(x_1, x_2) \mapsto x_i$, $i = 1, 2$, are measurable (i.e., for all $A_i \subseteq \mathcal{A}_i$ we have $p_i^{-1}(A) \in \mathcal{A}$.)
- $\mu = \mu_1 \otimes \mu_2$ is the unique measure on $\mathcal{A}$ such that $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ for $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$. (Existence: [Bauer Satz 23.3].)

Example 1. $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2}, \mathrm{Leb}_{\mathbb{R}^2}) = (\mathbb{R} \times \mathbb{R}, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}, \mathrm{Leb}_{\mathbb{R}} \otimes \mathrm{Leb}_{\mathbb{R}})$.

2. Let X be countable. Then $b : X \times X \to [0, \infty)$ is a measure via

$$b(A) = \sum_{(x,y)\in A} b(x, y), \quad A \subseteq X \times X.$$

Then, b is a product measure if and only if there are functions $b_1, b_2 : X \to [0, \infty)$ such that $b(x, y) = b_1(x)b_2(y)$.

Theorem 4. *(Fubini-Tonelli [Bauer, Satz 23.6, Korollar 23.7]) Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces. 1. (Tonelli) Let $f : X_1 \times X_2 \to [0,\infty]$ be measurable. Then,*

$$x_2 \mapsto \int_{X_1} f(\cdot, x_2) d\mu_1, \quad \text{and} \quad x_1 \mapsto \int_{X_2} f(x_1, \cdot) d\mu_2$$

are measurable and

$$\int_X f d\mu_1 \otimes \mu_2 = \int_{X_1} \int_{X_2} f(x_1, x_2) d\mu_2(x_2) d\mu_1(x_1) = \int_{X_2} \int_{X_1} f(x_1, x_2) d\mu_1(x_1) d\mu_2(x_2) \tag{$*$}$$

(Note that the equality includes the case that one and thus all terms are ∞.)

2. (Fubini) Let $f : X_1 \times X_2 \to \mathbb{C}$ be integrable. Then, $f(\cdot, x_2)$ (respectively $f(x_1, \cdot)$) are for μ_1-almost every x_1 (respectively μ_2-almost every x_2) integrable and the almost everywhere defined functions are

$$x_2 \mapsto \int_{X_1} f(\cdot, x_2) d\mu_1, \quad \text{and} \quad x_1 \mapsto \int_{X_2} f(x_1, \cdot) d\mu_2$$

integrable and we have $(*)$. In particular, f is integrable if either $\int_{X_1} \int_{X_2} |f| d\mu_2 d\mu_1 < \infty$ or $\int_{X_2} \int_{X_1} |f| d\mu_1 d\mu_2 < \infty$.

The following examples shows that one can not omit the exceptional sets of measure zero in the statement.

Counter-example Let $\mathrm{Leb}_{\mathbb{R}^2} = \mathrm{Leb}_{\mathbb{R}} \otimes \mathrm{Leb}_{\mathbb{R}}$ on $\mathbb{R}^2$ and $f = 1_{\mathbb{Q}\times\mathbb{R}}$. Then, by $(*)$ we have

$$\int_{\mathbb{R}} 1_{\mathbb{Q}\times\mathbb{R}} d\mathrm{Leb}_{\mathbb{R}^2} = \int_{\mathbb{R}} \int_{\mathbb{Q}} d\mathrm{Leb}_{\mathbb{R}} d\mathrm{Leb}_{\mathbb{R}} = \int_{\mathbb{R}} 0 d\mathrm{Leb}_{\mathbb{R}} = 0,$$

while $1_{\mathbb{Q}\times\mathbb{R}}(x, \cdot)$ is not integrable if $x \in \mathbb{Q}$.

Ende 3. Vorlesung

2.3 Operator theory

Literature: Joachim Weidmann "Lineare Operatoren in Hilberträumen I"

2.3.1 Hilbert spaces

Let V be a vector space. A *semi-scalar product* is an anti-linear map $s : V \times V \to \mathbb{K}$ that is linear in the second argument and positive on the diagonal (i.e., $s(f, g) = \overline{s(g, f)}$, $s(f, ag + h) = as(f, g) + s(f, h)$ and $s(f, f) \geq 0$ for $f, g, h \in V$ and $a \in \mathbb{K}$).

To a semi-scalar product we associate the corresponding *quadratic form* $q : V \to [0, \infty), f \mapsto q(f) := s(f, f)$. We will write $s(f) := q(f)$.

Facts:

- $s(af) = |a|^2 s(f)$ for $a \in \mathbb{K}$, in particular, $s(af) = s(f)$ if $|a| = 1$.
- By the polarization identity the sesqui-linear form is completely determined by its diagonal, i.e.,

$$s(f, g) = \frac{1}{4}\Big(s(f + g) - s(f - g) + is(f - ig) - is(f + ig)\Big)$$

- Cauchy-Schwarz(-Bunyakowski) inequality

$$|s(f, g)|^2 \leq s(f)s(g), \quad f, g \in V.$$

A semi-scalar product $s(\cdot,\cdot) = \langle \cdot,\cdot\rangle$ that is additionally positive definite (i.e., $\langle f,f\rangle = 0$ iff $f=0$) is called a *scalar product.* A scalar product defines a norm via

$$\|f\| = \langle f,f\rangle^{\frac{1}{2}}, \quad f\in V.$$

(The other direction is characterized by the Jordan/v.Neumann theorem via the parallelogram identity.)

The space V is called *complete* with respect to a scalar product $\langle\cdot,\cdot\rangle$ if it is complete as a metric space with respect to the metric $d(f,g) = \|f-g\| = \langle f-g, f-g\rangle^{\frac{1}{2}}$.

A Hilbert space is called *separable* if there is a countable basis that is a set of elements $e_\iota \in H$, $\iota\in J$ that is orthonormal, i.e., $\langle e_\iota, e_{\iota'}\rangle = 1_{\iota=\iota'}$ and for every $f\in H$ we have

$$f = \sum_{\iota\in J}\langle e_\iota, f\rangle e_\iota.$$

Examples 1. $\mathbb{K}^d$ with the Euclidian scalar product are Hilbert spaces with the standard Euclidian basis.

2. Let (X,m) be a discrete measure space (recall $m: X\to(0,\infty)$). Then

$$\langle f,g\rangle = \sum_{x\in X}\overline{f(x)}g(x)m(x)$$

defines a scalar product on

$$\ell^2(X,m) := \{f : X\to\mathbb{K} \mid \sum_{x\in X}|f(x)|^2 m(x) < \infty\}.$$

Indeed, $\ell^2(X,m)$ is complete and thus a Hilbert space and we can choose the functions $\delta_x = \frac{1}{\sqrt{m(x)}}1_{\{x\}}$, $x\in X$ as a basis (**Exercise 4**).

3. Let (M,μ) be a σ-finite measure space (e.g., $M\subseteq\mathbb{K}^d$). Then,

$$s(f,g) = \int_M \overline{f}g d\mu$$

defines a semi-scalar product on

$$\mathcal{L}^2(X,\mu) = \{f : M\to\mathbb{K} \mid |f|^2 \text{ is integrable, i.e., } \int_X |f|^2 d\mu < \infty\}.$$

It is not necessary a scalar product: If $f = 1_N$ for a set N of measure zero, then $\int_X f d\mu = \mu(N) = 0$ although $f\neq 0$. We can circumvent this issue by factoring out the functions that are zero almost everywhere: Let $\mathcal{N} = \{f \mid f\equiv 0\ \mu\text{-almost everywhere}\}$. Then, $\mathcal{N}$ is a subspace of $\mathcal{L}(M,\mu)$ and a function f is in $\mathcal{N}$ iff $s(f,f) = 0$. Define

$$L^2(M,\mu) := \mathcal{L}^2(M,\mu)/\mathcal{N} \quad (=\{f+\mathcal{N} \mid f\in L^2(M)\}).$$

That is we form equivalence classes: f and g are equivalent iff $f-g \in \mathcal{N}$. Thus, two elements of an equivalence class agree μ-almost everywhere. The addition, the scalar multiplication of these equivalence classes is defined via its representatives: i.e., if $[f]$, $[g]$ are equivalence classes and f and g representatives, then $a[f]+[g]=[af+g]$ for $a \in \mathbb{K}$ and we can define a scalar product by

$$\langle [f],[g]\rangle = s(f,g) = \int_M \overline{f} g d\mu.$$

It can be checked that $L^2(M,\mu)$ is complete and thus a Hilbert space. We will get used to write f for $[f]$.
Exercise 5: If (X,m) is a discrete measure space (i.e., $m : X \to (0,\infty)$) then, $\ell^2(X,m) = L^2(X,m)$.

2.3.2 Selfadjoint operators

Let H be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. Let $D \subseteq H$. We denote the closure of D in H with respect to the metric induced by metric $\langle\cdot,\cdot\rangle$ by $\overline{D} = \overline{D}^{\langle\cdot,\cdot\rangle}$. We say that D is *dense* in H if $\overline{D} = H$.

A *(linear) operator* on H is a linear mapping T from a subspace $D = D(T) \subseteq H$, i.e., $T(af+g) = aTf + Tg$, $f,g \in D$, $a \in \mathbb{K}$. We call $D = D(T)$ the *domain* of T. If $D(T)$ is dense in H then we say that T is *densely defined*.

An operator T is called positive if $\langle Tf, f\rangle \geq 0$ for all $f \in D(T)$.

Lemma 2. *Let T be a linear operator. The following are equivalent:*

(i) T is bounded*, i.e., there is $C \geq 0$ such that for every $f \in D$ with $\|Tf\| \leq C\|f\|$.*

(ii) T is continuous in every $f \in D$.

(iii) T is continuous in $0 \in D$.

Proof. (i) $\Rightarrow$ (ii): Let $\varepsilon > 0$ and $f \in D(T)$ be given. Choose $\delta = \varepsilon/C$. Then $\|Tf - Tg\| = \|T(f-g)\| \leq C\|f-g\| \leq \varepsilon$ for all $g \in B_\delta(f)$.
(ii) $\Rightarrow$ (iii): This is clear.
(iii)$\Rightarrow$ (i): Let $\varepsilon = 1$ and choose $\delta > 0$ accordingly. For arbitrary $f \neq 0$ set $g = \frac{\delta}{\|f\|} f$ which implies $\|g\| = \delta$. Hence, $\|Tf\| = \frac{\|f\|}{\delta}\|Tg\| \leq \frac{\|f\|}{\delta}\varepsilon = \frac{1}{\delta}\|f\|$. Thus, we have (i) with $C = 1/\delta$. □

If T is bounded and densely defined we can extend T uniquely to a bounded operator on H.

Examples 1. Let (M,μ) be a measure space, $H = L^2(M,\mu)$ and $V : M \to \mathbb{K}$ be a measurable function. Then, the operator M_V acting as

$$M_V f = Vf$$

on

$$D(M_V) = \{f \in L^2(M,\mu) \mid Vf \in L^2(\mathbb{R},\mu)\}$$

is a linear operator.
Exercise 6: $M_V^* = M_{\overline{V}}$, M_V is positive iff $V \geq 0$ μ-almost everywhere and bounded iff V is bounded μ-almost everywhere, i.e., f is in $L^\infty(\mathbb{R},\mu)$ which is the space of all μ-measurable functions $g : \mathbb{R} \to \mathbb{K}$ with $\|g\|_\infty = \inf\{C \geq 0 \mid |f(x)| \leq C$ μ-almost everywhere$\}$.

2. The operator $T = -\frac{d^2}{dx^2}$ on $C_c^2(\mathbb{R}) \subseteq L^2(\mathbb{R}, \mathrm{Leb})$.
Exercise 7: T is positive and unbounded.

Let T be an operator on H. Let

$$D^* = \{g \in H \mid \text{there is } h_g \in H \text{ such that } \langle h_g, f\rangle = \langle g, Tf\rangle \text{ for all } f \in D(T)\}.$$

Note that h_g is determined uniquely if T is densely defined, D^* is a subspace of H and the map $T^* : D^* \to H$, $g \mapsto h_g$ is a linear mapping (**Exercise 8**).

We call T^* the *adjoint* of T. A operator T is called *selfadjoint* if $D(T) = D(T^*)$ and $T = T^*$.

Examples 1. Every Hermitian $d \times d$ matrix A, (i.e., $A = A^*$) is a bounded selfadjoint operator on $\mathbb{C}^d$ with $D = \mathbb{C}^d$. Similarly, every symmetric real $d \times d$ matrix A, (i.e., $A = A^\top$) is a bounded selfadjoint operator on $\mathbb{R}^d$.

2. The operator M_V from above is selfadjoint iff V is real valued μ almost surely.

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2.3.3 Operators arising from forms

A semi-scalar product s_0 on a dense subspace D_0 of a Hilbert space H is sometimes referred to as a positive, symmetric sesqui-linear form. We call it simply a form.

To s_0 we associate the scalar product (**Exercise 9**)

$$\langle\cdot,\cdot\rangle_{s_0} = \langle\cdot,\cdot\rangle + s_0(\cdot,\cdot).$$

The form s is called *closable* if every $\|\cdot\|_{s_0}$-Cauchy sequence (f_n) with $\|f_n\| \to 0$ satisfies $\|f_n\|_{s_0} \to 0$. We denote by D the closure of D_0 with respect to $\langle\cdot,\cdot\rangle_{s_0}$. (Weidmann, Satz 1.37). Denote by s the extension of s_0 to $D = D(s)$. Moreover, D can be interpreted as a closed subspace of H with respect to the scalar product $\langle\cdot,\cdot\rangle_s = \langle\cdot,\cdot\rangle + s(\cdot,\cdot)$.

Examples Let $H = L^2(\mathbb{R},\mu)$ with $\mu = \mathrm{Leb}$ (real valued functions) and $D_0 = C_c^\infty(\mathbb{R})$.
Exercise 10*: $C_c^\infty(\mathbb{R})$ dense in $L^2(\mathbb{R},\mu)$.

1. Let

$$s_0(f,g) = \int f'(x)g'(x)d\mu.$$

(**Exercise 11**: s_0 is a closable form (semi-scalar product)). The closure $D(s)$ is the space of weakly differentiable functions f in $L^2(\mathbb{R},\mu)$ whose weak derivative is in $L^2(X,\mu)$ as well, i.e., to $f \in L^2(\mathbb{R},\mu)$ there is a function $\dot{f} \in L^2(\mathbb{R},\mu)$ with

$$\langle \dot{f}, g\rangle = -\langle f, g'\rangle \text{ for all } g \in C_c^\infty(\mathbb{R}).$$

Example: $\dot{f} = f'$ for $f \in C^1(\mathbb{R})$, **Exercise 12**: Let $H = L^2([-1,1],\mathrm{Leb})$. Compute $\dot{f}$ for $f : x \mapsto |x|$.
For the extension s, we have

$$s(f,g) = \int \dot{f}\dot{g}\,d\mu.$$

This (resp. the analogue in $\mathbb{R}^d$) is called the classical energy form (classical Dirichlet form).

2. Let $V : \mathbb{R} \to [0,\infty)$ be a $\mu =$ Leb measurable function and

$$r_0(g,h) = \int ghV\,d\mu.$$

Then, $D(r) = L^2(\mathbb{R},\mu) \cap L^2(\mathbb{R},V\mu) = \{f \in L^2(\mathbb{R},\mu) \mid \sqrt{V}f \in L^2(\mathbb{R},\mu)\}$. **Exercise 13**: r_0 is a closable form.

3. Mix Example 1 and 2, i.e.,

$$h_0(f,g) = (s_0 + r_0)(f,g) = \int (f'g' + Vfg)\,d\mu.$$

4. Let $H = L^2(\mathbb{R}^d,\mu)$, $D_0 = C_c^\infty(\mathbb{R}^d)$, $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ μ-measurable, symmetric and non-negative and ρ be a Radon measure.

$$\mathcal{E}_0^{(d)} = \int A\nabla f \cdot \nabla g\,d\mu + \int fg\,d\rho.$$

Theorem 5. *(Weidmann Abschnitt 4.2) Let s_0 be closable form on a dense subspace D_0 of a Hilbert space H. Then, there is a unique positive selfadjoint operator T with*

$$D(T) \subseteq D(s) \text{ dense w.r.t. } \langle\cdot,\cdot\rangle_s$$

and

$$\langle Tf, g\rangle = s_0(f,g), \quad f \in D_0 \cap D(T), g \in D_0.$$

Moreover,

$$D(T) = \{f \in D(s) \mid \text{ there exist } \hat{f} \in H \text{ such that } s(f,g) = \langle \hat{f}, g\rangle \text{ for } g \in D_0\},$$
$$Tf = \hat{f},\ f \in D(T).$$

In particular, $D(T)$ is dense in H if $D(s)$ is dense in H.

Idea: $s(f,g) = \langle T^{\frac{1}{2}}f, T^{\frac{1}{2}}g\rangle$ or integration by parts:

Examples. Let $L^2(\mathbb{R},\mu)$ with $\mu = $ Leb (real valued functions) and $D_0 = C_c^\infty(\mathbb{R})$.

1. $s_0(f,g) = \int f'(x)g'(x)d\mu$ as above. By the theorem, for $f \in D(T) \subseteq D(s)$ there is a function $\hat{f}$ such that for all $g \in C_c^\infty(\mathbb{R})$

$$\int \hat{f} g d\mu = \int \dot{f} g' d\mu = -\int f g'' d\mu.$$

Then, $D(T)$ is the set of functions $f \in L^2(\mathbb{R},\mu)$ such that there is function $\hat{f} = \ddot{f} \in L^2(\mathbb{R},\mu)$ such that

$$\int f g'' dx = (-\int \dot{f} g' dx =) \int \ddot{f} g dx$$

and

$$Tf = -\ddot{f}.$$

In particular, for $f \in C_c^2(\mathbb{R})$ we have $Tf = -f''$.

2. The operator from the form $r_0(f,g) = \int V f g d\mu$ of Example 2.b is given by $Tg = Vg$ and $D(T) = \{f \in L^2(\mathbb{R},\mu) \mid Vf \in L^2(\mathbb{R},\mu)\}$.

3. The operator from Example 3, $h_0(f,g) = \int (f'g' + fg)V d\mu$ is given by $Tf = -f'' + Vf$ for $f, g \in D_0$.

4. If $A = I$ and $\mu = $ Leb, then $Tf = -\Delta f + Vf$ for $f \in D_0$.

On the other hand if T is a positive selfadjoint operator. Then

$$s_0(f,g) = \langle Tf, g\rangle, \quad f, g \in D(T)$$

defines a closable form and the positive selfadjoint operator arising from the closure of s coincides with T. (**Exercise 14**.)

Corollary 1. *Let s_0 be a densely defined, positive, symmetric sesqui-linear form and T the corresponding operator. Then,*

$$\sup_{f,g\in D_0, \|f\|=\|g\|=1} |s_0(f,g)| = \sup_{f,g\in D(s), \|f\|=\|g\|=1} |s(f,g)| = \sup_{f\in D(T), \|f\|=1} \|Tf\|$$

In particular, s_0, s and T are all either bounded or unbounded.

Proof. Note first that

$$\begin{aligned}\|Tf\| &= \sup\{|\langle Tf, g\rangle| \mid f \in D(T), g \in H, \|f\| = \|g\| = 1\}\\ &= \sup\{|\langle Tf, g\rangle| \mid f, g \in D(T), \|f\| = \|g\| = 1\}\end{aligned}$$

since by Cauchy-Schwarz inequality $\|Tf\|\|g\| \geq |\langle Tf, g\rangle|$ we get '$\leq$' and with $g = \frac{1}{\|Tf\|}Tf$ we get $\|Tf\| = \langle Tf, g\rangle$ and, thus, '$\geq$'. The second statement follows since $D(T)$ is dense in H by the theorem above. Now, the equalities of the statements follow since D_0 and $D(T)$ are dense in $D(s)$. $\square$

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2.4 Dirichlet forms

Literature: Fukushima, Oshima, Takeda "Dirichlet Forms and symmetric Markov processes"

We now restrict our attention the case $H = L^2(M, \mu)$, where M is a locally compact second countable Hausdorff space and μ a Radon measure.

An important class of forms on $L^2(M, \mu)$ are so called Dirichlet forms. They measure the 'energy' of a function. They are characterized by being well compatible with so called normal contractions.

A map $C : \mathbb{R} \to \mathbb{R}$ is called a *normal contraction* if

$$C(0) = 0, \quad |C(u) - C(v)| \leq |u - v| \text{ for } u, v \in \mathbb{R}.$$

Examples. $u \mapsto |u|$, $u \mapsto u \wedge 1$, $u \mapsto u \vee 0$.

Let s be a form such that for every normal contraction $C : \mathbb{R} \to \mathbb{R}$ and every $f \in D(s)$

(C) $C \circ f \in D(s)$ and $s(C \circ f) \leq s(f)$ for $f \in D(s)$.

If s is additionally closed it is called a *Dirichlet form*. The important axiom is the second one which says that the 'energy becomes smaller' the less the function 'fluctuates'.

Exercise 15: Show that if a form satisfies the assumptions for C being $u \mapsto |u|$, $u \mapsto u \wedge 1$ and $u \mapsto u \vee 0$, then it is a Dirichlet form.

A Dirichlet form s is called *regular* if $D(s) \cap C_c(X)$ is dense in

- $D(s)$ with respect to $\| \cdot \|_s$
- $L^\infty(X)$ with respect to $\| \cdot \|_\infty$.

This means that the form can be approximated very well by compactly supported functions.

Theorem 6. *[Fukushima et al Theorem 3.1.1] Let s_0 be a closable form on $L^2(X, \mu)$ that satisfies (C). Then, the closure s is a Dirichlet form.*

The examples above are regular Dirichlet forms, see Fukushima et al. Ende 6. Vorlesung+Übung

Chapter 3

Forms and operators on graphs

3.1 Graphs

Let X be a countable set, $b : X \times X \to [0, \infty)$ be such that

(b1) $b(x, x) = 0$ for $x \in X$,

(b2) $b(x, y) = b(y, x)$ for $x, y \in X$,

(b3) $\sum_{z \in X} b(x, z) < \infty$ for $x \in X$,

and $c : X \to [0, \infty)$.

We can think of (b, c) as a graph over X in the following way: Let X be the set of vertices (nodes). Two vertices $x, y \in X$ are connected by an edge of weight $b(x, y)$ whenever $b(x, y) > 0$. In this case, we write for the edge and $x \sim y$. The weight can be thought as the conductivity, thickness or inverse length of an edge. We think of all vertices $x \in X$ with $c(x) > 0$ to have an one-way edge with weight $c(x)$ to a (imaginary) vertex at infinity.

We call a graph locally finite if the set $\{y \in X \mid b(x, y) > 0\}$ is finite for every $x \in X$.

Example Let $b : X \times X \to \{0, 1\}$ and $c \equiv 0$. Then, (b3) implies that (b, c) is locally finite. We call $(b, 0)$ an *unweighted graph.*

A *path* is sequence of vertices $(x_0, \ldots, x_n)$ with $x_{i-1} \sim x_i$, $i = 1, \ldots, n$. We say n is the *length* of the path. A graph is called *connected* if any two vertices can be connected by a path.

We define the following distance function d on X which we call the *natural graph metric (or distance).* Let $d(x, y)$ be the minimal n such that x and y can be connected by a path of length n. (**Exercise 16**: d is a metric iff (b, c) is connected.)

Moreover, the topology induced by d is the discrete topology (i.e., $\mathcal{O} = 2^X$). (**Exercise 17**.) Thus, every function $f : X \to \mathbb{R}$ is continuous (**Exercise 18**) and we denote the set of continuous functions by $C(X)$.

3.2 Borel measures on discrete spaces

Let X be countable and discrete (equipped with the discrete topology). The Borel σ-algebra over X is then 2^X. We know from the previous section that every σ-finite measure on X is given by a function $m : X \to [0, \infty)$ by letting

$$m(A) = \sum_{x \in A} m(x), \qquad A \subseteq X.$$

Additionally, we assume that m has full support, i.e., $m(x) > 0$ for all $x \in X$, (otherwise, replace X by $X' = \{x \in X \mid m(x) > 0\}$. In the case where a graph (b, c) over X is given one can also replace (b, c) by (b', c') which are defined as $b' = b|_{X' \times X'}$ and $c' = c|_{X'} + \sum_{y \in X \setminus X'} b(\cdot, y)$).

We call such a pair (X, m) a discrete measure space.

Example 1. $m \equiv 1$.
2. For a graph (b, c) over X and $n : X \to [0, \infty)$

$$n(x) = \sum_{y \in X} b(x, y) + c(x)$$

let $m = n$. If (b, c) is unweighted (i.e., $b(x, y) \in \{0, 1\}$ and $c \equiv 0$) then n is the vertex degree, i.e.,

$$n(x) = \deg(x) = \#\{y \in X \mid y \sim x\}$$

3.3 Function spaces

Let (X, m) be a discrete measure space. Then, the space

$$\ell^2(X, m) = \{f : X \to \mathbb{R} \mid \sum_{x \in X} |f(x)|^2 m(x) < \infty\}$$

equipped with the scalar product

$$\langle f, g \rangle = \sum_{x \in X} f(x) g(x) m(x) = \sum_{X} f g m$$

is a Hilbert space.
Lemma 3. *Convergence in $\ell^2(X, m)$ implies pointwise convergence.*

Proof. Let $f_n, f \in \ell^2(X,m)$, $n \in \mathbb{N}$ and $x \in X$ arbitrary. Then,

$$\|f_n - f\| = \Big(\sum_{x\in X} |f_n(x) - f(x)|^2 m(x)\Big)^{\frac{1}{2}} \geq |f_n(x) - f(x)| m(x)^{\frac{1}{2}}$$

which implies the statement. □

If $m \equiv 1$, we denote $\ell^2(X) := \ell^2(X,1)$.

Moreover, let $C_c(X)$ be the space of finitely supported functions, i.e.,

$$C_c(X) = \{f : X \to \mathbb{R} \mid \operatorname{supp} f := \{x \in X \mid f(x) \neq 0\} \text{ is finite}\}$$

which is obviously a subset of $\ell^2(X,m)$.
Exercise 19$C_c(X)$ is dense in $\ell^2(X,m)$.

Finally, let $\ell^\infty(X)$ be the space of bounded functions,

$$\ell^\infty(X) = \{f : X \to \mathbb{R} \mid \sup_{x\in X} |f(x)| < \infty\}.$$

Note that $\ell^\infty(X)$ does not depend on the choice of m.

Clearly, $C_c(X) \subset \ell^\infty(X)$. Moreover, if $\sum_{x\in X} m(x) < \infty$ then $\ell^\infty(X) \subseteq \ell^2(X,m)$. On the other hand, if $\inf_{x\in X} m(x) > 0$, then $\ell^2(X,m) \subseteq \ell^\infty(X)$.

Exercise 20: Find a counterexample such that neither $\ell^2(X,m)$ includes $\ell^\infty(X)$ nor vice versa.

3.4 Forms on graphs

Let (b,c) be a graph over a discrete measure space (X,m). Let $H = \ell^2(X,m)$ and $D_0 = C_c(X)$. We consider the form $Q^{(0)} = Q^{(0)}_{b,c}$ on D_0 given by

$$Q^{(0)}(f,g) = \frac{1}{2} \sum_{x,y\in X} b(x,y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x\in X} c(x) f(x) g(x),$$

for $f, g \in C_c(X)$. It will be shown in Lemma 4 that the sum converges absolutely and is a closable positive sesqui-linear form.

First, we motivate how $Q^{(0)}$ is a discrete analogue of continuum energy forms $\mathcal{E}(f,g) = \int (A\nabla f \cdot \nabla g + V f g) d\mu$ on $\mathbb{R}^d$.

3.4.1 Motivation of discrete analogue

So, we want to 'differentiate' functions on X. On $\mathbb{R}^d$, we say a function f is differentiable in $x \in \mathbb{R}^d$ in the direction $r \in \mathbb{R}$, $r \neq 0$, if

$$\partial_r f(x) = \lim_{h\to 0} \frac{f(x) - f(x+hr)}{h} = \lim_{h\to 0} \underbrace{\frac{f(x) - f(x+hr)}{|x - x + rh|}}_{=: d_{(x,x+rh)} f(x)} = \lim_{h\to 0} d_{(x,x+rh)} f(x)$$

exists. On a graph we can think of the directions from a vertex x as the pairs (x, y) for $x \neq y$. Indeed, only the (x, y) for y which have distance one from x will be relevant, i.e., (x, y) such that $b(x, y) > 0$. Hence, the difference quotient for a function $f \in C(X)$ can be written as

$$d_{(x,y)}f(x) = \frac{f(x) - f(y)}{d(x, y)} \underbrace{=}_{\text{if } x \sim y} f(x) - f(y)$$

(and $d_{(x,x)}f(x) = 0$). However, as there is only one point in the direction (x, y), this is the closest we can get to x. So, we consider $d_{(x,y)}f(x)$ as the directional derivative in the direction from x to y, i.e., '$\partial_{(x,y)}f(x) = d_{(x,y)}f(x)$'. Note that $d_{(x,y)}f(x) = -d_{(y,x)}f(y)$. (In this sense $C^1(X) = C(X)$.) Let $d : C(X) \to C(X \times X)$ be the linear operator given by

$$df(x, y) = d_{(x,y)}f(x).$$

Now we can consider b as a measure on $X \times X$ by $b(A \times B) = \sum_{x \in A, y \in B} b(x, y)$ and c as a measure on X by $c(A) = \sum_{x \in A} c(x)$. So we integrate the divergence term with respect to b and the potential term with respect to c, i.e.,

$$\begin{aligned} Q(f, g) &= \frac{1}{2} \int_{X \times X} df(x, y) dg(x, y) db + \int_X fg dc \\ &= \frac{1}{2} \sum_{x,y \in X} b(x, y)(d_{(x,y)}f(x))(d_{(x,y)}g(x)) + \sum_{x \in X} f(x)g(x)c(x). \end{aligned}$$

(Indeed, b has the density $(x, y) \mapsto b(x, y)/m(x)m(y)$ with respect to $m \otimes m$ and c has the density c/m with respect to m).

If $b(x, y) \in \{0, 1\}$, $c \equiv 0$ then

$$Q(f, g) = \frac{1}{2} \sum_{x \in X} \underbrace{\sum_{y \sim x} df(x, y) dg(x, y)}_{= \nabla f(x) \cdot \nabla g(x) = \sum_{i=1}^{d} \partial_i f(x) \partial_i g(x)} .$$

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3.4.2 Properties of $Q^{(0)}$

Define $\widetilde{Q} : \ell^2(X, m) \to [0, \infty]$ acting as

$$\widetilde{Q}(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f(x)^2$$

and

$$\widetilde{D} = \{f \in \ell^2(X, m) \mid \widetilde{Q}(f) < \infty\}.$$

Since $\widetilde{Q}$ satisfies the parallelogram identity on $\widetilde{D}$ we can extend $\widetilde{Q}$ to $\widetilde{D} \times \widetilde{D}$ by polarization. We define the scalar product

$$\langle \cdot, \cdot \rangle_{\widetilde{Q}} = \langle \cdot, \cdot \rangle + \widetilde{Q}(\cdot, \cdot)$$

and the corresponding norm by $\| \cdot \|_{\widetilde{Q}}$.

Lemma 4. *Let $Q^{(0)}$ be given as above.*

1. $|Q^{(0)}(f,g)| < \infty$ *for* $f, g \in C_c(X)$.
2. $Q^{(0)}$ *is a positive symmetric sesqui-linear form.*
3. $Q^{(0)}$ *is closable.*

Proof. The first item follows from (b3) and the Cauchy-Schwarz inequality, the second from non-negativity of (b2) and the the third item follows from Fatou's lemma.

Let us be more specific: 1. Let $f \in C_c(X)$ and $F = \mathrm{supp} f$. We estimate

$$Q^{(0)}(f) = \underbrace{\frac{1}{2} \sum_{x,y \in F} b(x,y)(f(x) - f(y))^2}_{<\infty, \text{ since } F \text{ finite}} + \underbrace{\sum_{x \in F} f(x)^2 \underbrace{\sum_{y \in X \setminus F} b(x,y)}_{<\infty, \text{ by (b3)}}}_{<\infty, \text{ since } F \text{ finite}} + \underbrace{\sum_{x \in F} c(x) f(x)^2}_{<\infty, \text{ since } F \text{ finite}} < \infty$$

By Cauchy-Schwarz inequality we see that for $f, g \in C_c(X)$

$$Q(f,g)^2 \leq Q(f)Q(g) < \infty.$$

2. Sesqui-linearity is clear. Symmetry follows from the symmetry of b, that is (b2). Positivity follows from non-negativity of b and c.

3. We first show that $\widetilde{Q}$ is closed, i.e. that the scalar product space $(\widetilde{D}, \| \cdot \|)$ is complete: Let $f_n \in \widetilde{D}$, $n \geq 1$, and $f \in \ell^2(X,m)$ with $f_n \to f$ pointwise or in ℓ^2. From Fatou's lemma it follows that $\widetilde{Q}$ is lower semi continuous, i.e.

$$\widetilde{Q}(f) = \widetilde{Q}(\liminf_{n \to \infty} f_n) = \liminf_{n \to \infty} \widetilde{Q}(f_n).$$

Hence, if $\liminf \widetilde{Q}(f_n) < \infty$ then $f \in \widetilde{D}$. Suppose (f_n) is a $\| \cdot \|_{\widetilde{Q}}$ Cauchy sequence. Since $\ell^2(X,m)$ is complete, there is $f \in \ell^2(X,m)$ such that $f_n \to f$ with respect to $\| \cdot \|$. It remains to show, that $f_n \to f$ with respect to $\| \cdot \|_{\widetilde{Q}}$. By Fatou's lemma

$$\widetilde{Q}(f - f_n) = \widetilde{Q}(\liminf_{k \to \infty} f_k - f_n) = \liminf_{k \to \infty} Q(f_k - f_n).$$

By the Cauchy sequence property the statement follows. □

3.4.3 The form Q

Let

$$D(Q) = \overline{C_c(X)}^{\|\cdot\|_{Q^{(0)}}}$$

and denote the restriction of $\widetilde{Q}$ to $D(Q) \times D(Q)$ by $Q = Q_{b,c}$.

Lemma 5. *Q is a regular Dirichlet form and*

$$Q(f) = \lim_{n\to\infty} Q^{(0)}(f_n),$$

where (f_n) are functions in $C_c(X) = D(Q^{(0)})$ that converge to $f \in D(Q)$ with respect to $\|\cdot\|_{\widetilde{Q}}$.

Proof. Let $C : \mathbb{R} \to \mathbb{R}$ be a normal contraction and $f \in C_c(X)$. Then, $C \circ f \in C_c(X)$ and $|C(f(x)) - C(f(y))| \leq |f(x) - f(y)|$ and $|C(f(x))| \leq |C(f(x))|$ for all $x, y \in X$ and $f \in C_c(X)$. Hence, $Q^{(0)}(C \circ f) \leq Q^{(0)}(f)$. Moreover, by Theorem 6 we have that the closure is a Dirichlet form.
Let (f_n) be functions in $C_c(X) = D(Q(^{(0)})$ that converge to $f \in D(Q)$ with respect to $\widetilde{Q}$. Then, for large n we get by Fatou's

$$|Q(f)^{\frac{1}{2}} - Q^{(0)}(f_n)^{\frac{1}{2}}| \leq \widetilde{Q}(f - f_n)^{\frac{1}{2}} + |\widetilde{Q}(f_n)^{\frac{1}{2}} - Q^{(0)}(f_n)^{\frac{1}{2}}| \leq \liminf_{k\to\infty} \widetilde{Q}(f_n - f_k)^{\frac{1}{2}}$$

which is small as (f_n) is a $\|\cdot\|_{\widetilde{Q}}$ Cauchy sequence. This finishes the proof. □

Example Let $b : X \times X \to \{0, 1\}$ and $c \equiv 0$. Then, by (b3) the graph (b, c) must be locally finite. Then, $Q^{(0)}$ on $D_0 = C_c(X)$ is given by

$$Q^{(0)}(f, g) = \frac{1}{2} \sum_{x,y\in X, x\sim y} (f(x) - f(y))(g(x) - g(y))$$

Lemma 6. *All regular Dirichlet forms on the measure space (X, m) are given by graphs (b, c) as $Q_{b,c}$.*

Proof. Let $\mathcal{E}$ be a regular Dirichlet form on $D(\mathcal{E}) \subseteq \ell^2(X, m)$. We have to show that there is a graph (b, c) such that $\mathcal{E} = Q_{b,c}$.
Step 1. $C_c(X) \subseteq D(\mathcal{E})$: Let $x \in X$ and $\varphi_x = 1_{\{x\}}$. If we show $\varphi_x \in D(\mathcal{E})$ then we are done as $D(\mathcal{E})$ is a vector space. Since $D(\mathcal{E}) \cap C_c(X)$ is dense in $(C_c(X), \|\cdot\|_\infty)$ there is $\psi \in D(\mathcal{E})$ such that $\psi(x) = 2$ and $\psi(y) < 1$ for $y \neq x$. Since $\mathcal{E}$ is a regular Dirichlet form $1 \wedge \psi \in D(\mathcal{E})$. It follows that $\varphi_x = \psi - (1 \wedge \psi) \in D(\mathcal{E})$.

Step 2. $\mathcal{E}(\varphi_x, \varphi_y) \leq 0$ for $x \neq y$: For $x \neq x$ let $u = \varphi_x - \varphi_y$ and note that $|u| = \varphi_x + \varphi_y$. Since $\mathcal{E}$ is a Dirichlet we have $\mathcal{E}(|u|) \leq \mathcal{E}(u)$ and thus

$$\mathcal{E}(\varphi_x) + 2\mathcal{E}(\varphi_x, \varphi_y) + \mathcal{E}(\varphi_y) \leq \mathcal{E}(\varphi_x) - 2\mathcal{E}(\varphi_x, \varphi_y) + \mathcal{E}(\varphi_y)$$

which implies the statement.

Step 3. $\sum_{y\in X}\mathcal{E}(\varphi_x,\varphi_y)\in[0,\infty)$ for $x\in X$: Fix $x\in X$. By Step 2. we have that

$$\sum_{y\in X}\mathcal{E}(\varphi_x,\varphi_y)\le\mathcal{E}(\varphi_x)<\infty.$$

On the other hand, let $K\subseteq X$ be finite with $x\in X$ and let $\varepsilon>0$. Set $u=1_K+\varepsilon\varphi_x$ and note that $1\wedge u=1_K$. Since $\mathcal{E}$ is a Dirichlet we have $\mathcal{E}(1\wedge u)\le\mathcal{E}(u)$ and, thus,

$$\mathcal{E}(1_K)\le\mathcal{E}(1_K)+2\varepsilon\mathcal{E}(\varphi_x,1_K)+\varepsilon^2\mathcal{E}(\varphi_x).$$

This implies

$$\sum_{y\in K}\mathcal{E}(\varphi_x,\varphi_y)=\mathcal{E}(\varphi_x,1_K)\ge-\frac{\varepsilon}{2}\mathcal{E}(\varphi_x)\to 0$$

as $\varepsilon\to 0$. On the other hand, exhausting X by compact sets K we get the statement of Step 3.

Set $b(x,y)=-\mathcal{E}(\varphi_x,\varphi_y)$ for $x\neq y$, $b(x,x)=0$ and $c(x)=\sum_{x,y}\mathcal{E}(\varphi_x,\varphi_y)$. Moreover, every $\psi\in C_c(x)$ can be represented as $\psi=\sum_{x\in X}\psi(x)\varphi_x$ and we get

$$\begin{aligned}\mathcal{E}(\psi)&=\sum_{x,y\in X}\psi(x)\psi(y)\mathcal{E}(\varphi_x,\varphi_y)\\&=\frac{1}{2}\sum_{x,y\in X,x\neq y}\mathcal{E}(\varphi_x,\varphi_y)(\psi(x)-\psi(y))^2+\sum_{x\in X}\varphi(x)^2\sum_{y\in X}\mathcal{E}(\varphi_x,\varphi_y)\\&=\frac{1}{2}\sum_{x,y\in X}b(x,y)(\psi(x)-\psi(y))^2+\sum_{x\in X}c(x)\varphi(x)^2.\end{aligned}$$

Since $C_c(X)$ is dense in $D(\mathcal{E})$ with respect to $\|\cdot\|_{\mathcal{E}}$ the statement follows by polarization. □

3.5 Integrated Leibniz rule

On $\mathbb{R}^d$ there is the Leibniz rule

$$\nabla fg(x)=f\nabla g(x)+g\nabla f(x).$$

For the discrete case one has

$$\begin{aligned}(fg(x)-fg(y))&=f(x)(g(x)-g(y))+g(y)(f(x)-f(y))\\&=f(x)(g(x)-g(y))+g(x)(f(x)-f(y))-(f(x)-f(y))(g(x)-g(y))\end{aligned}$$

However, we have an integrated Leibniz rule.

Lemma 7. *(Leibniz rule) Let $f, g, h \in C(X)$. Then,*

$$\sum_{x,y\in X} b(x,y)(fg(x) - (fg)(y))(h(x) - h(y))$$
$$= \sum_{x,y\in X} b(x,y)f(x)(g(x) - g(y))(h(x) - h(y)) + \sum_{x,y\in X} b(x,y)g(x)(f(x) - f(y))(h(x) - h(y)),$$

whenever two of the terms converge absolutely.

Proof. The statement follows directly from the first formula above by renaming x and y in the second sum. □

Define
$$d_b f(x,y) = b(x,y)^{\frac{1}{2}} df(x,y) = b(x,y)^{\frac{1}{2}}(f(x) - f(y))$$
for $f \in C(X)$ and $d_b f \cdot d_b g$ by

$$(d_b f \cdot d_b g)(x) = \sum_{y\in X} d_b f(x,y) d_b g(x,y) = \sum_{y\in X} b(x,y)(f(x) - f(y))(g(x) - g(y))$$

whenever the sum converges absolutely. Then, the Leibniz rule reads as

$$\sum_X d_b(fg) \cdot d_b h = \sum_X f d_b g \cdot d_b h + \sum_X g d_b f \cdot d_b h$$

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3.6 Selfadjoint operators

In the previous section we learned that $Q^{(0)}$ is closable and in the previous Toolbox section we discussed how to extract a self adjoint operator from a closable form. In this section we will show that this operator L is a restriction of the *formal Laplacian* $\mathcal{L}$ given as

$$\mathcal{L}f(x) = \frac{1}{m(x)} \sum_{y\in X} b(x,y)(f(x) - f(y)) + \frac{1}{m(x)} c(x)f(x),$$

defined on

$$\mathcal{F} = \{f \in C(X) \mid \sum_{y\in X} b(x,y)|f(y)| < \infty \text{ for all } x \in X\}.$$

(In some sense one can consider '$\mathcal{F} = C^2(X)$'). Obviously, $C_c(X) \subseteq \mathcal{F}$ and if (b,c) is locally finite, then $\mathcal{F} = C(X)$.

Theorem 7. *There is a selfadjoint operator L with domain $D(L)$ dense in $\ell^2(X,m)$*

$$Q(f,g) = \langle Lf, g\rangle, \quad f \in D(L), g \in D(Q)$$

which is a restriction of $\mathcal{L}$, i.e., L is acting as

$$Lf(x) = \frac{1}{m(x)} \sum_{y\in X} b(x,y)(f(x) - f(y)) + \frac{1}{m(x)} c(x)f(x), \quad f \in D(L), x \in X.$$

Moreover,

$$D(L) = \{f \in \ell^2(X,m) \mid \text{ there is } \hat{f} \in \ell^2(X,m) \text{ such that } Q(f,g) = \langle \hat{f}, g\rangle \text{ for all } g \in C_c(X)\}$$

Proof. By Theorem 5 from Section 2.3.3 there is a densely defined selfadjoint operator L such that $Q^{(0)}(f,g) = \langle Lf, g\rangle$ for $f \in D(L), g \in D(Q^{(0)})$. As Q is the closure of $Q^{(0)}$ we also have $Q(f,g) = \langle Lf, g\rangle$ for $f \in D(L), g \in D(Q)$. Moreover, for $x \in X$ let $\varphi_x = 1_{\{x\}}/m(x)$. Clearly $\varphi_x \in C_c(X) \subseteq D(Q)$. Then, we have for $f \in D(L)$

$$\begin{aligned} Lf(x) &= \langle Lf, \varphi_x\rangle = Q(f, \varphi_x) \\ &= \frac{1}{2m(x)} \sum_{y,z\in X} b(y,z)(f(y) - f(z))(1_{\{x\}}(y) - 1_{\{x\}}(z)) + \frac{1}{m(x)} \sum_{y\in X} c(y)f(y)1_{\{x\}}(y) \\ &= \frac{1}{m(x)} \sum_{y\in X} b(x,y)(f(x) - f(y)) + \frac{1}{m(x)} c(x)f(x) \end{aligned}$$

As the left hand side is finite, so is the right hand side and, in particular, the sum $\sum_{y\in X} b(x,y)f(y)$ converges absolutely. Thus, $D(L) \subseteq \mathcal{F}$. □

To determine $D(L)$ is usually not an easy question. In particular, we will show below that in general not even $C_c(X) \subseteq D(L)$.

By let us give two canonical examples first.

Examples Let $b : X \times X \to \{0,1\}$ and $c \equiv 0$. Then, $Q^{(0)}(f) = \frac{1}{2}\sum_{x\sim y}(f(x) - f(y))^2$. Define the (vertex) degree $\deg : X \to \mathbb{N}_0$ by

$$\deg(x) = \sum_{y\in X} b(x,y) = \#\{y \sim x\}.$$

1. Let $m = 1$. Denote the closure of $Q^{(0)}$ in $\ell^2(X)$ by Q_1, i.e. the closure with respect to $\|f\|_{Q_1}^2 = \sum_X f^2 + Q^{(0)}(f)$. Then, the corresponding operator $L = \Delta$ acts as

$$\Delta f(x) = \sum_{y\sim x} (f(x) - f(y)), \quad f \in D(\Delta).$$

One calls Δ the Laplacian.

2. Let $m = \deg$. Denote the closure of $Q^{(0)}$ in $\ell^2(X, \deg)$ by $Q_{\deg}$, i.e., the closure with respect $\|f\|^2_{Q_{\deg}} = \sum_X f^2 \deg + Q^{(0)}(f)$. Then, the corresponding operator $L = \widetilde{\Delta}$ acts as

$$\widetilde{\Delta} f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} (f(x) - f(y)), \quad f \in D(\widetilde{\Delta})$$

and $\widetilde{\Delta}$ is called the *normalized Laplacian*. We will see next that $\widetilde{\Delta}$ is a bounded operator.

3.7 Boundedness

The following theorem characterizes when the operator L is bounded and since $D(L)$ is dense in $\ell^2(X, m)$ it follows $D(L) = \ell^2(X, m)$ (**Exercise 21**). Define *weighted degree* $\mathrm{Deg} : X \to [0, \infty)$

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \Big(\sum_{y \in X} b(x, y) + c(x) \Big).$$

Theorem 8. *(Boundedness) The following are equivalent:*

(i) Deg *is bounded,*

(ii) Q *is bounded, i.e.,* $\sup_{f,g \in D(Q), \|f\| = \|g\| = 1} |Q(f, g)| < \infty$,

(iii) L *is bounded on* $\ell^2(X, m)$,

(iv) $\mathcal{L}$ *is bounded on* $\ell^\infty(X)$.

In this case, Q, L *and* $\mathcal{L}|_{\ell^\infty}$ *are bounded by* $2 \sup_{x \in X} \mathrm{Deg}(x)$.

Proof. (i)⇒(ii): As $(\alpha + \beta)^2 \le 2\alpha^2 + 2\beta^2)$

$$\begin{aligned} Q(f) &= \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f(x)^2 \\ &\le \frac{1}{2} \sum_{x,y \in X} b(x, y) f(x)^2 + \frac{1}{2} \sum_{x,y \in X} b(x, y) f(y)^2 + \sum_{x \in X} c(x) f(x)^2 \\ &\le 2 \sum_{x \in X} \mathrm{Deg}(x) f(x)^2 m(x) \\ &\le 2d \|f\|^2, \end{aligned}$$

where $d := \sup_{x \in X} \mathrm{Deg}(x)$. The statement for $Q(f, g)$ follows from polarization.
(ii)⇒(i): Let $\delta_{x'} = 1_{\{x'\}} / \sqrt{m(x')}$. Then,

$$Q(\delta_{x'}) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(\delta_{x'}(x) - \delta_{x'}(y))^2 + \sum_{x \in X} c(x) \delta_{x'}(x)^2 = \mathrm{Deg}(x).$$

(ii)⇔(iii): Follows from Corollary 1 in Section 2.3.3.

(i)⇒(iv): The statement follows as for $f \in \ell^\infty(X)$ we can estimate using $|f(x) - f(y)| \leq 2\|f\|_\infty$

$$|\mathcal{L}f(x)| \leq 2\|f\|_\infty \frac{1}{m(x)} \Big| \sum_{y \in X} b(x,y) + c(x) \Big| = 2\|f\|_\infty \mathrm{Deg}(x).$$

(iv)⇐(i): Using $1_{\{x\}}$ we see that $\mathcal{L}1_{\{x\}}(x) = \mathrm{Deg}(x)$ and the theorem follows. □

Examples Let $b : X \times X \to \{0, 1\}$ and $c \equiv 0$. Recall $\deg(x) = \#\{y \sim x\}$.

1. For $\Delta f(x) = \sum_{y \sim x}(f(x) - f(y))$ on $\ell^2(X)$ we get that Δ is bounded iff deg is bounded as Deg = deg in this case.

2. For $\widetilde{\Delta} f(x) = \frac{1}{\deg(x)} \sum_{y \sim x}(f(x) - f(y))$ on $\ell^2(X, \deg)$ we get that $\widetilde{\Delta}$ is always bounded by 2 as $\mathrm{Deg}(x) = \frac{1}{d(x)} \sum_{y \sim x} 1 = 1$, $x \in X$, in this case.

3.8 Green's formula

The classical Green formula for $\Omega \subseteq \mathbb{R}^d$ open with smooth boundary reads as

$$\int_\Omega \nabla f \cdot \nabla g = \int_\Omega (\Delta f) g - \int_{\partial\Omega} g(\nabla f \cdot \nu) = \int_\Omega f(\Delta g) - \int_{\partial\Omega} f(\nabla g \cdot \nu).$$

We want to give a discrete analogue for $\Omega = X$. We are particulary interested in the case when the boundary terms vanish. Indeed, the various phenomena depend on the fact that this is not always the case. To this end we leave ℓ^2 for a while and consider a larger universe of functions which we can put into Q.

For $f, g \in C(X)$ define

$$\mathcal{Q}(f, g) = \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x) f(x) g(x)$$

whenever the sum converges absolutely.

Assume that $f, g \in C(X)$ satisfy the following assumptions

$$\sum_{x,y \in X} b(x,y)|f(x)||g(y)| < \infty, \quad \sum_{x,y \in X} b(x,y)|f(x)||g(x)| < \infty, \quad \sum_{x \in X} c(x)|f(x)||g(x)| < \infty. \tag{Q}$$

Lemma 8. *Let $f, g \in C(X)$. If f, g satisfies (Q) then the sum of $\mathcal{Q}$ converges absolutely. If $f \in \mathcal{F}$ and $g \in C_c(X)$, then f, g satisfy (Q).*

Proof. The first statement is clear. For the second statement let $f \in \mathcal{F}$ and $g \in C_c(X)$. Then,

$$\sum_{x,y\in X} |b(x,y)f(x)g(y)| = \sum_{y\in X} |g(y)| \underbrace{\sum_{x\in X} b(x,y)|f(y)|}_{<\infty,\, f\in\mathcal{F}} < \infty$$

and

$$\sum_{x,y\in X} |b(x,y)f(x)g(x)| = \sum_{x\in X} |f(x)||g(x)| \underbrace{\sum_{y\in X} b(x,y)}_{<\infty, \text{ by (b3)}} < \infty.$$

Finally, $\sum c|fg| < \infty$ as it is a finite sum since $g \in C_c(X)$. $\square$

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Lemma 9. *(Green's formula) Let $f, g \in \mathcal{F}$ satisfy (Q). Then,*

$$\mathcal{Q}(f,g) = \sum_X (\mathcal{L}f)gm = \sum_X f(\mathcal{L}g)m.$$

and all three sums converge absolutely. This is in particular the case if $f \in \mathcal{F}$ and $g \in C_c(X)$.

Remark: One should think of the statement as $\int \nabla f \cdot \nabla g dx = \int (\Delta f) g dx = \int f(\Delta g) dx$.

Proof. By the assumption that the sums converge absolutely we have that $|\mathcal{Q}(f,g)|$ can be estimated by these three sums and thus converges. Moreover,

$$\begin{aligned}
&\frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c(x)f(x)g(x) \\
=&\frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))g(x) - \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))g(y) + \sum_{x\in X} c(x)f(x)g(x)
\end{aligned}$$

where the first two sums converge absolutely by (Q) and the triangle inequality. Thus, we continue to calculate

$$\begin{aligned}
\ldots &= \sum_{x,y\in X} b(x,y)(f(x)-f(y))g(x) + \sum_{x\in X} c(x)f(x)g(x) \\
&= \sum_{x\in X} (\mathcal{L}f(x))g(x)m(x).
\end{aligned}$$

The other equality follows analogously. $\square$

3.9 $C_c(X) \subseteq D(L)$

In this section we address the question whether the compactly supported functions are included in the domain of L.

Theorem 9. *The following are equivalent:*

(i) The functions $\psi_x : X \mapsto \mathbb{R}$, $y \mapsto \frac{b(x,y)}{m(y)}$ *are in* $\ell^2(X,m)$ *for all* $x \in X$.

(ii) $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$.

(iii) $C_c(X) \subseteq D(L)$.

This is, in particular, the case if (b,c) *is locally finite or* $\inf_{x\in X} m(x) > 0$.

Proof. (i)⇔(ii): The statement $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ is equivalent to $\mathcal{L}1_{\{x\}} \in \ell^2(X,m)$ for all $x \in X$ (since $C_c(X) = \mathrm{lin}\{1_{\{x\}} \mid x \in X\}$). We have

$$\mathcal{L}1_{\{x\}}(y) = \begin{cases} \frac{1}{m(x)}\left(\sum_{z\in X} b(x,z) + c(x)\right) & y = x, \\ -\frac{1}{m(y)}b(x,y) = -\psi_x(y) & y \neq x. \end{cases}$$

Thus,

$$\|\mathcal{L}1_{\{x\}}\|^2 = \underbrace{\frac{1}{m(x)^2}\Big|\sum_{z\in X} b(x,z) + c(x)\Big|^2 m(x)}_{=:C_x<\infty} + \sum_{y\in X\setminus\{x\}} \frac{b(x,y)^2}{m(y)^2} m(y) = C_{x'} + \|\psi_{x'}\|^2.$$

Hence, $\mathcal{L}1_{\{x\}} \in \ell^2(X,m)$ iff $\psi_x \in \ell^2(X,m)$.

(ii)⇒(iii): By Theorem 7 we have to show that for all $\varphi \in C_c(X)$ there is $\widehat{f} \in \ell^2(X,m)$ such that $Q(\varphi,g) = \langle \widehat{f}, g\rangle$ for all $g \in C_c(X)$. Set $\widehat{f} = \mathcal{L}\varphi$. By assumption $\mathcal{L}\varphi \in \ell^2(X,m)$ and by Greens formula, Lemma 9, (which is applicable as $\varphi, g \in C_c(X) \subseteq \mathcal{F}$)

$$Q(\varphi,g) = \sum_X (\mathcal{L}\varphi) g m = \langle \mathcal{L}\varphi, g\rangle.$$

Thus, $\varphi \in D(L)$.

(iii)⇒(ii): This is clear as L is a map $D(L) \to \ell^2(X,m)$ and $\mathcal{L}f = Lf$ for $f \in D(L)$ by Lemma 7.

If (b,c) is locally finite, then $\psi_x \in C_c(X)$ for all $x \in X$. If $c := \inf_{x\in X} m(x) > 0$, then

$$\|\psi_x\|^2 = \sum_{y\in X} \frac{b(x,y)^2}{m(y)} \leq \frac{1}{c}\sum_{y\in X} b(x,y)^2 \leq \frac{1}{c}\Big(\underbrace{\sum_{y,b(x,y)\geq 1} b(x,y)^2}_{<\infty,\text{ finite sum by (b3)}} + \underbrace{\sum_{y,b(x,y)<1} b(x,y)}_{<\infty,\text{ by (b3)}} \Big) < \infty.$$

(or as one also can say $\ell^1(X) \subseteq \ell^2(X)$.)

□

Example 1.For the operator Δ, the graph (b,c) is always locally finite. Thus, $\psi_x : y \mapsto \frac{b(x,y)}{m(y)}$ is compactly supported and we have $C_c(X) \subset D(\Delta)$ by the theorem above.

2. Since $\widetilde{\Delta}$ is bounded on $\ell^2(X, \deg)$ by Theorem 8, we trivially have $D(\widetilde{\Delta}) = \ell^2(X, \deg) \supseteq C_c(X)$.

3. Finally, we give an example of a graph (b,c) over a (X,m) with $C_c(X) \not\subseteq D(L)$. Let $X = \mathbb{N}_0$, $b(0,n) = b(n,0) = \frac{1}{n^2}$ and $b(n,n') = 0$ otherwise and $c \equiv 0$. Thus, (b3) is satisfied. Choose m such that $m(0) = 1$ and $m(n) = \frac{1}{n^4}$. Thus, ψ_0 from Theorem 9 is given by $\psi_0(n) = n^2$ is not in $\ell^2(X,m)$.

Chapter 4

Toolbox B. Spectrum of operators

4.1 The spectrum

Let T be an operator on a Hilbert space H. Spectral theory is motivated by the study of solutions of the equation

$$(T - zI)f = g$$

for given $g \in H$ and $z \in \mathbb{C}$. Desirable properties of the solution $f \in D(T)$ are:

- Existence of a solution for all g, ($T - z$ onto)
- Uniqueness of the solution, ($T - z$ one-to-one)
- Continuity of the solution with respect to g, ($(T - z)^{-1}$ is continuous)
- Continuity of the solution with respect to z, (if (i), (ii), (iii) are satisfied)

Define the resolvent set $\rho(T)$ as

$$\rho(T) = \{z \in \mathbb{C} \mid (T - z) : D(T) \to H \text{ is bijective and } (T - z)^{-1} \text{ is continuous}\}.$$

Moreover, we call its complement set

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

the spectrum of T (as Hilbert did).

Examples 1. Let $A \in \mathbb{C}^{d\times d}$ on $\mathbb{C}^{d\times d}$. Then, $(A - z)$ is bijective iff z is no eigenvalue and in this case $(A - z)^{-1}$ is continuous. Hence,

$$\sigma(A) = \{z \in \mathbb{C} \mid \text{there is } u \neq 0 \text{ such that } Au = zu\} = \{\text{eigenvalues of } A\}.$$

2. Let $V : \mathbb{R}^d \to \mathbb{R}$ continuous and let M_V be the operator on $L^2([0, 1]^d, \mathrm{Leb})$ given by

$$D(M_V) = \{f \in L^2([0, 1]^d, \mathrm{Leb}) \mid Vf \in L^2([0, 1]^d, \mathrm{Leb})\}, \quad M_V f = Vf.$$

Since $M_{\frac{1}{V-z}}$ is the inverse of M_{V-z} for all $z \notin \text{ran}V := \{V(x) \mid x \in [0,1]^d\}$, we have

$$\sigma(M_V) = \text{ran}V.$$

3. Let (X, μ) be a σ-finite measure space and $V : X \to \mathbb{R}$ be a μ-measurable function. Let M_V be given as above

$$D(M_V) = \{f \in L^2(X,\mu) \mid Vf \in L^2(X,\mu)\}, \quad M_V f = Vf.$$

Then,

$$\sigma(M_V) = \text{ran}_\mu(V) := \{E \in \mathbb{R} \mid \mu(V^{-1}(E-\varepsilon, E+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}.$$

This example shows that the spectrum of a multiplication is easy to analyze. In the next section we will see that every selfadjoint operator can be represented as a multiplication operator.

Ende 10. Vorlesung

4.2 The spectral theorem and its consequences

The spectral theorem is the ticket to a better mathematical life.

Let $(H_n, \langle\cdot,\cdot\rangle_n)$, $n \in \mathbb{N}$, be Hilbert spaces. Then,

$$H = \bigoplus_{n\in\mathbb{N}} H_n = \{(f_n) \mid f_n \in H_n \text{ and } \sum_n \|f\|_n^2 < \infty\}$$

is a Hilbert space with respect to the scalar product $\langle f, g\rangle = \sum_{n\in\mathbb{N}} \langle f_n, g_n\rangle_n$ for $f = (f_n), g = (g_n) \in H$, Moreover, if T_n are selfadjoint operators on H_n and P_n the orthogonal projection $H \to H_n$ (i.e., $P_n = P_n^*$ and $P_n^2 = I$), then there exists a selfadjoint operator T on H such that

$$D(T) = \{f \in H \mid P_n f \in D_n, \sum_{n\in\mathbb{N}} \|T_n P_n f\|^2 < \infty\},\ Tf = (T_n P_n f_n)$$

and $\sigma(T) = \bigcup_{n\in\mathbb{N}} \sigma(T_n)$. (**Exercise 22**).

Example. Let (M_n, μ_n), $n \in \mathbb{N}$ be measure spaces. Then, there is a measure space (M, μ) such that $L^2(M,\mu) = \bigoplus_{n\in\mathbb{N}} L^2(M_n, \mu_n)$ (**Exercise 23**, choose the σ-algebra generated by the compact sets).

Let H_1, H_2 be Hilbert spaces. An operator $U : H_2 \to H_1$ is called a *unitary* operator if $U^*U = UU^* = I$, i.e., $U^{-1} = U^*$.

Theorem 10. *(Spectral theorem for selfadjoint operators) Let T be a selfadjoint operator on a separable Hilbert space H. Then, there is a σ-finite measure space (M, μ) and a measurable function $V : M \to \mathbb{R}$ such that T is unitarily equivalent to M_V on $L^2(M,\mu)$, i.e., there is a unitary operator*

$$U : L^2(M,\mu) \to H \text{ with } T = U M_{V\text{id}} U^{-1}.$$

Picture: commutative diagram

Remark Indeed (M,μ) can be chosen such that $L^2(M,\mu) = \bigoplus_{n\in\mathbb{N}} L^2(\mathbb{R},\mu_n)$ where μ_n are Radon measures and V is the identity function $\mathrm{id} : x \mapsto x$ on the copies.

Unfortunately it is usually very hard (impossible) to determine U.

Examples 1. Let $A \in \mathbb{R}^{d\times d}$ be symmetric on $\mathbb{R}^d$, $u_1,\ldots,u_d$ be an orthonormal basis of eigenfunctions to the eigenvalues $E_1,\ldots,E_d$. For $U = (u_1,\ldots,u_d)$ we have $U^*U = UU^* = I$ and $UAU^* = \mathrm{diag}(E_1,\ldots,E_d)$. If all E_j have multiplicity one then let $\mu = \sum_{n=1}^d \delta_{E_n}$, where δ_x is the point measure at x and $V = \mathrm{id}$. If the multiplicity is higher take respectively more copies.

2. The Laplace operator $-\Delta$ on $L^2(\mathbb{R}^d,\mathrm{Leb})$ is unitarily equivalent via Fourier transform to $V : \mathbb{R}^d \to \mathbb{R}, k \mapsto |k|^2$ on $L^2(\mathbb{R}^d,\mathrm{Leb})$

Corollary 2. $\sigma(T) = \{E \in \mathbb{R} \mid \mu(V^{-1}(E-\varepsilon,E+\varepsilon)) > 0$ *for all* $\varepsilon > 0\}(=\mathrm{ran}_\mu V)$. *In particular, if* $V = \mathrm{id}$*, then* $\mu(\mathbb{C}\setminus\sigma(T)) = 0$.

We get the following corollary.

Corollary 3. *Let s be a positive, symmetric closed sesquilinear form and T the corresponding selfadjoint operator from Theorem 5. Then,*

$$\begin{aligned}\inf\sigma(T) &= \inf\{s(f,f) \mid f \in D(s), \|f\| = 1\} = \inf\{s_0(f,f) \mid f \in D_0, \|f\| = 1\}\\ \sup\sigma(T) &= \sup\{s(f,f) \mid f \in D(s), \|f\| = 1\} = \sup\{s_0(f,f) \mid f \in D_0, \|f\| = 1\}.\end{aligned}$$

Proof. Let$M = \bigotimes_{n\geq 1}\mathbb{R}$, $\mu = \bigotimes_{n\geq 1}\mu_n$ and $U : H \to L(M,\mu)$ be the unitary operator such that $U^*TU = M_{\mathrm{id}}$. Then,

$$\sigma(T) = \sigma(M_{\mathrm{id}}).$$

For $\varphi \in L^2(M,\mu) = \bigoplus_{n\geq} L^2(\mathbb{R},\mu_n)$ denote by φ_n the component of φ on $L^2(\mathbb{R},\mu_n)$. We calculate,

$$\begin{aligned}\inf\sigma(M_{\mathrm{id}}) = \inf\operatorname{supp}\mu &= \inf_{\varphi\in L^2(M,\mu),\|\varphi\|=1}\ \inf_{n\geq 1}\int_{\mathbb{R}} t\varphi_n(t)^2 d\mu_n(t)\\ &= \inf_{\varphi\in L^2(M,\mu),\|\varphi\|=1}\langle M_{\mathrm{id}}\varphi,\varphi\rangle = \inf_{\varphi\in L^2(M,\mu),\|\varphi\|=1}\langle UTU^*\varphi,\varphi\rangle\\ &= \inf_{\varphi\in L^2(M,\mu),\|\varphi\|=1}\langle TU^*\varphi,U^*\varphi\rangle = \inf_{f\in H,\|f\|=1}\langle Tf,f\rangle.\end{aligned}$$

By density of D_0 and $D(T)$ in $D(s)$ with respect to $\|\cdot\|_s$ the statement follows from the corollary above. □

The spectral theorem allows is to define functions of selfadjoint operators T. This is referred to as the functional calculus. Let a measurable function $\varphi : \sigma(T) \to \mathbb{C}$ be given and define

$$\varphi(T) := UM_{\varphi\circ V}U^{-1}$$

i.e.,

$$D(\varphi(T)) := UD(M_{\varphi\circ V}), \quad \varphi(T)Uf = UM_{\varphi\circ V}f.$$

Facts:

- $\varphi(T)^* = \overline{\varphi}(T)$
- $\varphi(T)$ is bounded if φ.

Examples 1. For a symmetric matrix A with eigenvectors $U = (u_1, \dots, u_d)$ and eigenvalues $E_1, \dots, E_d$ one defines $\varphi(A) = U\mathrm{diag}(\varphi(E_1), \dots, \varphi(E_d))U^{-1}$.

2. For the Laplacian $-\Delta$ and the Fourier transform F define $\varphi(\Delta) = FM_{\varphi(|\cdot|^2)}F^{-1}$.

Clearly, multiplication by a characteristic function of a measurable set is a projection on L^2 (i.e., an idempotent selfadjoint operator). we now can easily deducible following proposition from the spectral theorem and the properties of multiplication operators.

Proposition 1. *Then $1_A(T)$ is an orthogonal projection for every measurable $A \subseteq \mathbb{R}$. The map*

$$E : \mathcal{B} \to \textit{orthogonal projections}, \quad E(A) = 1_A(T)$$

is a projection valued measure, i.e.,

$$E(\dot{\bigcup}_{j\in\mathbb{N}} A) = \bigoplus_{j\in\mathbb{N}} E(A_j), \quad A_j \in \mathcal{B} \textit{ with } A_j \cap A_k = \emptyset, j \neq k$$

that is $E(A_k)E(A_j) = E(A_j)E(A_k) = 0$ for $j \neq k$ and $\| \sum_{j=1}^n E(A_j)f - E(A)f\| \to 0$, $n \to \infty$, for all $f \in H$ and

$$E(\mathbb{R}) = I \quad \textit{and} \quad E(\emptyset) = 0.$$

Moreover, we have

$$\sigma(T) = \operatorname{supp} E := \{\lambda \in \mathbb{R} \mid E(\lambda - \varepsilon, \lambda + \varepsilon) \neq 0 \textit{ for all } \varepsilon > 0\}.$$

Proof. **Exercise 24** □

The mapping E is called the *spectral family* of T. It is unique in contrast to the mapping U above. By E and $f \in H$ we can define a the spectral measure with of f with respect to T by

$$\mu_f(A) = \langle E(A)f, f\rangle = \|E(A)f\|^2, \quad A \in \mathcal{B}.$$

It has the following fundamental property.

Proposition 2. *Let $\varphi : \sigma(T) \to \mathbb{C}$ be measurable. Then $f \in D(\varphi(T))$ iff $\varphi \in L^2(\mathbb{R}, \mu_f)$. In this case $\|\varphi(T)f\|^2 = \int |\varphi|^2 d\mu_f$*

Proof. **Exercise 25**. □

4.3 Semigroups, resolvents and characterization of Dirichlet forms

Two particular important examples of functions φ are $x \mapsto \frac{1}{t-\lambda}$ for $\lambda \notin \sigma(T)$ and $x \mapsto e^{-tx}$, $t \geq 0$ which yield the resolvent $(T-\lambda)^{-1}$ and the semigroup e^{-tT}.

Indeed, for $T = -\Delta$ we see that given $f \in L^2(\mathbb{R}^d)$, $\lambda < 0$, the resolvent $g = (\Delta+\lambda)^{-1}$ solves the Poisson equation

$$(\Delta + \lambda)g = f$$

and the semigroup $\varphi_t = e^{-t\Delta} f$ solves the heat equation

$$\Delta \varphi_t = \partial \varphi_t, \quad \varphi_0 = f.$$

Semigroups and resolvents are connected by the following important formula.

Lemma 10. *Let T be a selfadjoint operator on a Hilbert space H. For $\lambda < \inf \sigma(T)$ we have*

$$(T-\lambda)^{-1} f = \int_0^\infty e^{\lambda t} e^{-tT} f dt, \quad f \in H,$$

where the integral on the right hand side is a Riemann integral.

Proof. For $\lambda < \lambda_0 = \inf \sigma(T)$ the identity

$$\frac{1}{x-\lambda} = \int_0^\infty e^{-(x-\lambda)t} dt$$

holds for $x \geq \lambda_0$. By the spectral theorem and Fubini's theorem

$$\begin{aligned} \langle f, (L-\lambda)^{-1} f \rangle &= \int_{\lambda_0}^\infty \frac{1}{x-\lambda} d\mu_f(x) \\ &= \int_{\lambda_0}^\infty \int_0^\infty e^{-xt} e^{-\lambda t} dt d\mu_f(x) \; (= \langle f, \int_0^\infty e^{-\lambda t} e^{-t\Delta} dt f \rangle) \\ &= \int_0^\infty e^{-\lambda t} \int_{\lambda_0}^\infty e^{-xt} d\mu_f(x) dt \\ &= \int_0^\infty e^{-\lambda t} \langle f, e^{-t\Delta} f \rangle dt. \end{aligned}$$

By polarization

$$\langle g, (L-\lambda)^{-1} f = \int_0^\infty e^{\lambda t} \langle g, e^{-tT} f dt, \quad f, g \in H$$

and the statement follows. □

The importance of regular Dirichlet forms rises from the fact that their resolvents and semi-groups have particulary nice properties.

Let s be a positive closed form s on $L^2(M,\mu)$. Recall that s is called a Dirichlet forms if for all $f \in D(s)$ we have $C \circ f \in D(s)$ and $s(C \circ f) \leq s(f)$ for all normal contractions $C : \mathbb{R} \to \mathbb{R}$. Let T be the selfadjoint operator arising from s.

A function $f : M \to \mathbb{R}$ is called *positive* if $f(x) \geq 0$ for almost all $x \in M$ and $f \not\equiv 0$. It is called *strictly positive* if $f(x) > 0$ for almost all $x \in M$ and we write $f > 0$. Accordingly, for functions f, g, h we write $f \leq g$ if $g - f$ is positive.

Theorem 11. *(Second Beurling-Deny criteria) The following are equivalent.*

(i) *s is a Dirichlet form.*

(ii) $0 \leq e^{-tT} f \leq 1$, $t > 0$, *for* $f \in L^2(M,\mu)$ *with* $0 \leq f \leq 1$.

(iii) $0 \leq \alpha(T+\alpha)^{-1} f \leq 1$, *for* $f \in L^2(M,\mu)$ *with* $0 \leq f \leq 1$.

The equivalence (ii)⇒(iii) follows from the lemma above. For proof see Fukushima, Oshima, Takeda 'Dirichlet Forms and symmetric Markov processes' Theorem 1.4.1.

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Corollary 4. *If* $e^{-tT} f \geq 0$, $t \geq 0$, *for* $f \in L^2(M,\mu)$ *with* $f \geq 0$. *Then,* $(T-\lambda)^{-1} f \geq 0$, *for all* $\lambda < \inf \sigma(T)$ *and* $f \in L^2(M,\mu)$ *with* $f \geq 0$.

Proof. By the lemma above

$$(L-\lambda)^{-1} f = \int_0^\infty e^{\lambda t} e^{-tT} f dt.$$

□

4.4 Weyl sequences

Stollmann 'Caught by disorder' Lemma 1.4.4 and Proposition 4.1.10.

Theorem 12. *Let s be closed, positive, symmetric form and let T be the associated selfadjoint operator on a Hilbert space H. Then the following are equivalent*

(i) $\lambda \in \sigma(T)$,

(ii) *There are* $u_n \in D(T)$ *with* $\|u_n\| = 1$, $n \in \mathbb{N}$, *and*

$$\|(T-\lambda)u_n\| \to 0, \quad n \to \infty,$$

(iii) *There are* $v_n \in D(s)$ *with* $\|v_n\| = 1$, $n \in \mathbb{N}$, *and*

$$\sup_{w \in D(s), s(w)+\|w\|^2 \leq 1} |(s-\lambda)(v_n, w)| \to 0, \quad n \to \infty,$$

and $(s-\lambda)(u,v) = s(u,v) - \lambda\langle u, v\rangle$.

Lemma 11. *Let $\lambda \in \mathbb{R}$. If for some $C > 0$ all*

$$\|f\| \leq C\|(T-\lambda)f\|, \quad \text{for all } f \in D(T)$$

then λ is in the resolvent set.

Proof. By the assumption it follows that λ is not an eigenvalue. Then, $\operatorname{Ker}(T-\lambda) = \operatorname{Ran}(T-\lambda) = \{0\}$, since $\operatorname{Ran}(A)^\perp = \operatorname{Ker}(A)$ for selfadjoint densely defined A. We show that $T - \lambda$ is bijective, i.e., $\operatorname{Ran}(T-\lambda) = H$. Then, the assumption implies $\|(T-\lambda)^{-1}\| \leq C$ and thus $\lambda \notin \sigma(T)$.
Let $g \in H$ and $f_n \in D(T)$ such that $g_n = (T-\lambda)f_n \to 0$ (which exists as $\operatorname{Ran}(T-\lambda)^\perp = \{0\}$ and thus $\operatorname{Ran}(T-\lambda)$ dense in H). By the assumption, both (f_n) and (Tf_n) are Cauchy and since

$$\langle f, Th \rangle = \lim_{n\to\infty} \langle f_n, Th \rangle = \lim_{n\to\infty} \langle Tf_n, h \rangle = \langle f, g \rangle$$

we have $f \in D(T)$ and $g = (T-\lambda)f$. □

Proof. The statement is clear if λ is an eigenvalue, so assume λ is no eigenvalue.

(i)⇒(ii): By the previous lemma there is a sequence (g_n) in $D(T)$ such that

$$\|f_n\| \geq n\|(T-\lambda)f_n\|$$

Letting $u_n := \frac{1}{\|f_n\|} f_n$ we get the statement.
(ii)⇒(iii): Let $v_n \in D(s)$ such that $\|u_n - v_n\|_s \leq 1/n$. Let $w \in D(s)$, $\|g\|_s = 1$. Then,

$$\begin{aligned}|(s-\lambda)(v_n, w)| &\leq |(s-\lambda)(u_n, w)| + |(s-1)(v_n - u_n, w)| + (|\lambda| + 1)|\langle v_n - u_n, w\rangle| \\ &\leq \|(T-\lambda)u_n\|\|w\| + \|u_n - v_n\|_s\|w\|_s + (\|\lambda| + 1)\|u_n - v_n\|\|w\| \to 0.\end{aligned}$$

(iii)⇒(i): Assume $\lambda \notin \sigma(T)$ and (v_n) as in (iii). Then,

$$c := \sup_{n\geq 1} \|(T-\lambda)^{-1} v_n\|_s < \infty$$

This gives

$$1 = \|v_n\|^2 = (s-\lambda)(v_n, (T-\lambda)^{-1}v_n) \leq c \sup_{w\in D(s), \|w\|_s = 1} (s-\lambda)(v_n, w) \to 0,$$

a contradiction. □

4.5 Compact operators and essential spectrum

A subset of a topological space is called *relatively compact* if its closure is compact. In particular, if we are in a complete metric space then a set is relatively compact iff every sequence in this set has a Cauchy subsequence. (**Exercise 26**.)

The following theorem characterizes the compact operators on a separable Hilbert space. The proofs can be found Weidmann 'Lineare Operatoren in Hilberträumen I' Kapitel 3.

Theorem 13. *(Compact operators) Let K be a bounded selfadjoint operator on a separable Hilbert space H. Then, the following are equivalent:*

(i) K maps bounded sets to relatively compact sets.

(ii) $\sigma(K) = \{\lambda_n\}_{n\geq 0}$ where (λ_n) converges to zero. In particular, if $m_n \in \mathbb{N}$ is the multiplicity of $\lambda_n \in \sigma(L)$ and $\psi_1^{(n)}, \ldots, \psi_{m_n}^{(n)}$ are orthonormal eigenfunctions to λ_n then

$$K = \sum_{\lambda_n \in \sigma(K)} \lambda_n P_n, \quad P_n = \sum_{j=1}^{m_n} \langle \psi_j^{(n)}, \cdot \rangle \psi_j^{(n)}.$$

in the sense such that $\|K - \sum_{k=1}^{n} \lambda_k P_k\| \to 0$, $n \to \infty$.

(iii) K is the norm limit of finite dimensional operators.

(iv) If $(f_n) \in D(K)$ converges weakly to zero, then (Kf_n) converges in the norm, (i.e., if $\langle f_n, \psi\rangle \to 0$ for all $\psi \in H$, then $\|Kf_n\| \to 0$).

Let us turn to the definition of the essential spectrum. This can be considered as a very stable under small perturbations. The proofs can be found Weidmann 'Lineare Operatoren in Hilberträumen I' Kapitel 8 and 9

Theorem 14. *(Essential spectrum) Let T be a selfadjoint operator on a separable Hilbert space H and $\lambda \in \mathbb{R}$. Then, the following are equivalent:*

(i) $\lambda \in \sigma(T)$ and λ is no isolated eigenvalue of finite multiplicity.

(ii) There is a weak null-sequence (f_n) of normalized vectors in $D(T)$ such that $\|(T-\lambda)f_n\| \to 0$.

(iii) $E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}$ has infinite dimensional range for all $\varepsilon > 0$ (where $E_x = 1_{(-\infty,x]}(T)$).

(iv) $\lambda \in \sigma(T+K)$ for all compact self adjoint operators K (where the sum is defined via the quadratic forms).

Proof. The equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) are found in Weidmann, Satz 8.24 and (i),(ii),(iii)$\Rightarrow$(iv) Satz 9.14. Thus it remains to show to one direction.

(iv)$\Rightarrow$(ii): As compact operators are bounded we have $D(T) = D(T+K)$ for compact K. By Weyl's criterion there are sequences $(f_n^{(K)})$ in $D(T)$ with $\|f_n\| = 1$ and $\|(T+K-\lambda)f_n^{(K)}\| \leq 1/n$ for compact K. Let $\{e_n\}$ be a basis of H included in $D(T)$. Let P_n be the orthogonal projection on $\operatorname{span}\{e_1, \ldots, e_{n-1}\}$. Hence, $I - P_n$ is finite dimensional and thus $-TP_n$ is compact. Let $f_n = (I-P_n)f_n^{(-TP_n)}$. Clearly $\|f_n\| = 1$. Moreover,

$$\|Tf_n\| = \|T(I-P_n)f_n^{(-TP_n)}\| = \|T - TP_n f_n^{(-TP_n)}\| \leq \frac{1}{n}$$

For $\varphi \in H$ we get by Cauchy Schwarz

$$|\langle \varphi, f_n\rangle| = |\langle \varphi, (I-P_n)f_n^{(-TP_n)}\rangle| = |\langle (I-P_n)\varphi, f_n^{(-TP_n)}\rangle| \leq \|(I-P_n)\varphi_n\| \to \infty,$$

as $n \to \infty$. Thus. we finished the proof. $\square$

We denote the set of all λ which satisfy the assumptions above by $\sigma_{\text{ess}}(T)$ and call it the essential spectrum of T. Moreover, denote

$$\lambda_0(T) = \inf \sigma(T), \quad \lambda_0^{\text{ess}}(T) = \inf \sigma_{\text{ess}}(T)$$

Since, $\sigma(T) \subseteq \sigma_{\text{ess}}(T)$

$$\lambda_0(T) \leq \lambda_0^{\text{ess}}(T).$$

Proposition 3. *Let s be a closed positive quadratic form on a separable Hilbert space H and let T be the corresponding selfadjoint operator. Assume there is a normalized sequence (f_n) in $D(s)$ that converges weakly to zero. Then,*

$$\lambda_0^{\text{ess}}(T) \leq \liminf_{n\to\infty} s(f_n)$$

Proof. The statement is clear for $\lambda_{0^{\text{ess}}}(T) = 0$. Let $\lambda < \lambda_0^{\text{ess}}$. We show $s(f_n) > \lambda$ for large n. Let λ_1 be such that $\lambda < \lambda_1 < \lambda_0^{\text{ess}}$ and let $\varepsilon > 0$ be arbitrary. Since, $D(T)$ is dense in $D(s)$ with respect to $\|\cdot\|_s$ there is a g_n for all $n \geq 0$ such that, $\|f_n - g_n\|_s^2 = s(f_n - g_n) + \|f_n - g_n\|^2 < \varepsilon$ and (g_n) converges weakly to zero as well. As $\lambda_1 < \lambda_0^{\text{ess}}$, the spectral projection $E_{\lambda_1} = 1_{(-\infty,\lambda_1]}(T)$ is a finite rank operator. Therefore, as (g_n) converges weakly to zero, there is $N \geq 0$ such that $\|E_{\lambda_1} g_n\|^2 < \varepsilon$ for $n \geq N$. For the spectral measure $\mu_n = \mu_{g_n}$ of T with respect to g_n, we estimate for $n \geq N$

$$h(g_n) \geq \int_{\lambda_1}^{\infty} t d\mu_n(t) \geq \lambda_1 \int_{\lambda_1}^{\infty} d\mu_n(t) = \lambda_1(\|g_n\|^2 - \|E_{\lambda_1} g_n\|^2) > \lambda_1(1-\varepsilon),$$

where we used $\lambda_1 \geq 0$ as $s \geq 0$. Since $s(f_n) \geq s(g_n) - \varepsilon$ by the choice of g_n, we conclude the asserted inequality by the choosing $\varepsilon = (\lambda_1 - \lambda)/(1 + \lambda_1) > 0$. □

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Proposition 4. *Let s_n, $n \in \mathbb{N}$, be closed positive quadratic forms on a separable Hilbert space H and let T_n be the corresponding selfadjoint operators. Assume*

- *$D(s_{n+1}) \subseteq D(s_n)$,*
- *the operators K_n arising from $s_0 - s_n$ are compact,*
- *all sequences (f_n) with $f_n \in D(s_n)$ are weak null-sequences.*

Then,

$$\lambda_0^{\text{ess}}(T_0) = \lim_{n\to\infty} \lambda_0(T_n)$$

Proof. By assumption and Theorem 14 (iv) we have $\lambda_0^{\text{ess}}(T_0) = \lambda_0^{\text{ess}}(T_0 + K_n) = \lambda_0^{\text{ess}}(T_n)$ for all $n \in \mathbb{N}$. As $\lambda_0(T_n) \leq \lambda_0^{\text{ess}}(T_n)$ we have

$$\limsup_{n\to\infty} \lambda_0(T_n) \leq \lambda_0^{\text{ess}}(T_n).$$

On the other hand, let $f_n \in D(s_n)$ such that $s(f_n) \leq \lambda_0(T_n) + 1/n$. By assumption (f_n) is a weak null-sequence the proposition above, we have

$$\lambda_0^{\text{ess}}(T_n) \leq \liminf_{n\to\infty} s(f_n) \leq \liminf_{n\to\infty} \lambda_0(T_n).$$

□

4.5.1 Application to graphs

Let (b,c) be a graph over (X,m). Let $U \subseteq X$ and Q_U be the closure of the restriction of Q to $C_c(U) \subseteq C_c(X)$ (by continuation by zero). Then, $D(Q_U) \subseteq D(Q)$. Let L_U be the corresponding selfadjoint operator and $\mathcal{L}_U$ the corresponding formal Laplacian with domain $\mathcal{F}_U$. Then, the following holds, **Exercise 27**

- If $b_U := b \cdot 1_{U\times U}$ and c_U given as $c_U(x) = c(x) + \sum_{y\in X\setminus U} b(x,y)$ for $x \in U$ and $c \equiv 0$ otherwise, then Q_U is the form which arises from (b_U, c_U) on (X,m)
- $\mathcal{F} \subset \mathcal{F}_U$.
- $\mathcal{L}_U = \mathcal{L}$ for $f \in \mathcal{F}$ with $\operatorname{supp} f \subseteq U$, in particular, if f is also in $D(L_U) \cap D(L)$ then $L_U = L$.

If the graph is locally finite and $K \subseteq X$ is a finite subset of X, then the operator $L - L_{X\setminus K}$ is finite dimensional and, thus ,compact. Therefore,

$$\lambda_0^{\mathrm{ess}}(L) = \lambda_0^{\mathrm{ess}}(L_{X\setminus K}).$$

If (K_n) is an exhausting sequence (that is $K_n \subseteq K_{n+1}$ and $X = \bigcup_n K_n$) of finite sets, then Proposition 4 above gives for locally finite graphs

$$\lambda_0^{\mathrm{ess}}(L) = \lim_{n\to\infty} \lambda_0(L_{X\setminus K_n}).$$

For general exhausting sequences (K_n) and general graphs we still have by Proposition 3

$$\lambda_0^{\mathrm{ess}}(L) \leq \liminf_{n\to\infty} \lambda_0(L_{X\setminus K_n}).$$

Exercise 28: The local finiteness assumption can be replaced by the assumption $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$.

Exercise 29*: Show that if (b,c) is locally finite then $D(L_U) = \{f \in D(L) \mid \operatorname{supp} f \subseteq U\}$? What happens in the general case?

Chapter 5

Positive solutions and spectrum

In the following we study the relation of properties of solutions and the spectrum. In this chapter and later in Chapter 7 will prove the following two results

- An Allegretto-Piepenbrink type theorem which characterizes the regime below bottom of spectrum by the existence of positive solutions
- A Shnol' type theorem which characterizes the spectrum by the existence of slowly growing solutions.

Such results can be shown in various continuum models such as Schrödinger operators on $\mathbb{R}^d$, Riemannian manifolds, strongly local Dirichlet forms etc. and we will prove them in the context of graphs here. In this chapter we treat the case of positive solutions.

Let us recall the setting from the first chapter.

Let (b, c) be a graph satisfying (b1), (b2) and (b3) (i.e., $b(x, x) = 0$, $b(x, y) = b(y, x)$ and $\sum_z b(x, z) < \infty$, $x, y, z \in X$) over a discrete measure space (X, m),

$$Q(f) = \frac{1}{2} \sum_{x \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f(x)^2$$

a form with $D(Q) = \overline{C_c(X)}^{\|\cdot\|_Q} \subseteq \ell^2(X, m)$ and L be the corresponding selfadjoint operator which is a restriction of $\mathcal{L}$ on $\mathcal{F} = \{f \in C(X) \mid \sum_{y \in X} b(x, y)|f(y)| < \infty, x \in X\}$ acting as

$$\mathcal{L}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x).$$

Let

$$\lambda_0 = \inf \sigma(L).$$

Let $\lambda \in \mathbb{R}$ and $U \subseteq X$. We call $u : X \to \mathbb{R}$ a solution (respectively a super-solution) to λ on U if $u \in \mathcal{F}$ and

$$\mathcal{L}u(x) = \lambda u(x), \quad \text{(respectively } \mathcal{L}u(x) \geq \lambda u(x)), \quad x \in U.$$

If $U = X$, then we call u a solution (resp. super-solution) to λ.

In the context of graph u positive means $u(x) \geq 0$ for all $x \in X$ and $u \not\equiv 0$ (since m has full support) and u strictly positive, $u > 0$, means $u(x) > 0$ for all $x \in X$.

We aim for the following result.

Theorem 15. *If (b, c) is infinite, connected and locally finite (i.e., $\#\{y \sim x\} < \infty$ for all x), then the following are equivalent:*

(i) $\lambda \leq \lambda_0$

(ii) There exists a positive super-solution to λ, (i.e., there is $u : X \to [0, \infty)$ with $0 \not\equiv u \in \mathcal{F}$ and to $\mathcal{L}u \geq \lambda u$).

(iii) There exists a strictly positive solution to λ, (i.e., there is $u : X \to (0, \infty)$ with $u \in \mathcal{F}$ and to $\mathcal{L}u = \lambda u$).

For general (b, c) we still have the equivalence (i)⇔(ii).

What about (i)⇒(iii) in general?

For finite graphs this is clearly wrong since the only solutions are the eigenvectors of L. For non locally finite graph we give the following counterexample:

Example Let $X = \mathbb{N}_0$, $m \equiv 0$ and (b, c) a star graph: $b(k, n) > 0$ iff k or n are zero and $c \equiv 0$. Let u be a positive solution to $\lambda \neq 0$. Then, for $k > 0$

$$\mathcal{L}u(k) = b(k, 0)(u(k) - u(0)) = \lambda u(k)$$

and

$$\mathcal{L}u(k) = \sum_{k=1}^{\infty} b(0, k)(u(0) - u(k)) = \lambda u(0).$$

Summing the first equation over k and adding both equations yields

$$\lambda(u(0) + \sum_{k=1}^{\infty} u(k)) = 0.$$

Let us sketch the idea of the proof:

(i)⇒(ii)/(iii): Resolvents are positive super-solutions: $g_x^{(\lambda)} = (L - \lambda)^{-1}\delta_x$ satisfies $(L - \lambda)g_x^{(\lambda)} \geq 0$. Let $x \to \infty$ to get a solution to λ and let $\lambda \to \lambda_0$ to get a solution to λ_0

(ii)⇒(i): Ground state transform
For a solution $u > 0$ to $\lambda \leq \lambda_0$ there is a positive form Q_u such that

$$Q_u(f) = Q(f) - \lambda \|f\|, \quad f \in C_c(X)$$

Ende 13. Vorlesung

5.1 A Harnack inequality

The Harnack inequality gives bounds for the growth of super-solutions.+

Theorem 16. *(Harnack inequality) Let $K \subseteq X$ be finite and connected. Then, for every positive $\lambda \in \mathbb{R}$ and every super-solution u to λ on K there is $C_K(\lambda)$ such that*

$$\max_{x \in K} u(x) \leq C_K(\lambda) \min_{x \in K} u(x)$$

Moreover, the function $\lambda \mapsto C_K(\lambda)$ continuous and monotone decreasing.
Remark. As we shall see later there are no super-solutions to $\lambda > \lambda_0$. So, we will only apply the statement for $\lambda \leq \lambda_0$.

Proof. Let $u \geq 0$ be a positive super-solution to λ on K. Clearly u is also a super-solution to all $\lambda' \leq \lambda$ (as $\mathcal{L}u \geq \lambda u \geq \lambda' u$ by $u \geq 0$). Let $I \subseteq \mathbb{R}$ be the maximal interval such that there exists a positive super-solution on K to all values in I.

Let $K \subseteq X$ finite, $\lambda \in I$ and u a super-solution to λ on K. Let $x_{\max}, x_{\min} \in K$ be the vertices where u takes its maximum/minimum in K. Let $(x_0, \ldots, x_n)$ be a path from $x_{\max}$ to $x_{\min}$. Employing $(\mathcal{L} - \lambda)u(x_j) \geq 0$

$$\begin{aligned} 0 &\leq \frac{1}{m(x_j)} \sum_{y \in X} b(x,y)(u(x_j) - u(y)) + \Big(\frac{c(x_j)}{m(x_j)} - \lambda\Big) u(x_j) \\ &\leq \Big(\underbrace{\frac{1}{m(x_j)} \sum_{y \in X} b(x,y) + \frac{c(x_j)}{m(x_j)}}_{=\mathrm{Deg}(x_j)} - \lambda \Big) u(x_j) - \frac{b(x_j, x_{j+1})}{m(x_j)} u(x_{j+1}). \end{aligned}$$

since $\sum_{y \neq x_{j+1}} b(x_j, y) u(y) \geq 0$ follows from $u \geq 0$. Hence,

$$u(x_{j+1}) \leq \frac{m(x_j)}{b(x_j, x_{j+1})} \Big(\mathrm{Deg}(x_j) - \lambda\Big) u(x_j).$$

and

$$u(x_{\max}) \leq \prod_{j=0}^{n-1} \frac{m(x_j)}{b(x_j, x_{j+1})} \Big(\mathrm{Deg}(x_j) - \lambda\Big) u(x_{\min})$$

Thus, the statement follows with

$$C_K(\lambda) := \max_{x,y \in K} \min_{x = x_0 \sim \ldots \sim x_n \sim y} \prod_{j=0}^{n-1} \frac{m(x_j)}{b(x_j, x_{j+1})} \Big(\mathrm{Deg}(x_j) - \lambda\Big).$$

Clearly C_K is continuous and monotone decreasing on I and we can extend it in this way to $\mathbb{R}$. □

Remark. The proof already shows that for a connected graph there are no positive super-solutions to $\lambda \geq \inf_x \mathrm{Deg}(x)$.

The Harnack inequality immediately gives a pointwise bound for positive super-solutions.

Corollary 5. *Let (b, c) be connected, $I \subseteq \mathbb{R}$ bounded and $x_0 \in X$. Then, there is a function $C := C_{x_0}(I) : X \to (0, \infty)$ such that for every positive super-solution $u \geq 0$ to $\lambda \in I$ we have*

$$C^{-1}(x)u(x_0) \leq u(x) \leq C(x)u(x_0).$$

In particular, every positive super-solution is strictly positive.

Proof. For $x \in X$ fix a path $(x_0, \ldots, x_n)$ from x_0 to x. Let $K = \{x_0, \ldots, x_n\}$ and $C(x) = C_{x_0}(I)(x) = \sup_{\lambda \in I} C_K(\lambda)$ (which exists as C_K is monotone decreasing). Then, by the Harnack inequality we obtain

$$u(x) \leq \max_{i=0,\ldots n} u(x_i) \leq C(x) \min_{i=0,\ldots n} u(x_i) \leq C(x)u(x_0)$$
$$u(x_0) \leq \max_{i=0,\ldots n} u(x_i) \leq \min_{i=0,\ldots n} u(x_i) \leq C(x)u(x).$$

□

Remark. The corollary shows that the space of super-solutions to λ in an bounded interval is compact with respect to the topology of pointwise convergence.

5.2 Convergence of (super-)solutions

Lemma 12. *Let $x_0 \in X$ and $\lambda \leq \lambda_0$,*

- *(λ_n) be a sequence of real numbers in $(-\infty, \lambda_0]$ converging to λ.*
- *$X_n \subseteq X$, $x_0 \in X_n \subseteq X_{n+1}$, $X = \bigcup_{n \in \mathbb{N}} X_n$ connected.*
- *u_n with $u_n(x_0) = 1$ be positive super-solutions to λ_n on X_n, $n \in \mathbb{N}$.*

Then, there is (n_k) and a strictly positive super-solution $u \in \mathcal{F}$ such that

$$u(x) = \lim_{k \to \infty} u_{n_k}(x), \quad \textit{for all } x \in X.$$

Moreover, if the graph is locally finite and u_n are solutions to λ_n on X_n, then u is a solution to λ on X.

Proof. We enumerate the vertices of X, i.e., $X = \{x_l \mid l \in \mathbb{N}_0\}$ such that $x_l \in X_l$. We define u inductively (with respect to l) via defining subsequences $(n_k^{(l)})$: Let $n_k^{(0)} = k$, $k \in \mathbb{N}$. Suppose we found subsequences $(n_k^{(l)}) \subseteq \ldots \subseteq (n_k^{(1)})$ such that the sequences $(u_{n_k^{(l)}}(x_l)), \ldots, (u_{n_k^{(1)}}(x_1)), (u_{n_k^{(0)}}(x_0))$ converge. For large k (i.e. such that $n_k^{(l)} \geq l$ the function $u_{n_k^{(l)}}$ is a solution to $\lambda_{n_k^{(l)}}$ on $X_{n_k^{(l)}}$. By Corollary 5 that

$$C(x_{l+1})^{-1} \leq u_{n_k^{(l)}}(x_{l+1}) \leq C(x_{l+1}), \quad \text{for } k \text{ large.}$$

Thus, there is a subsequence $(n_k^{(l+1)}) \subseteq (n_k^{(l)})$ such that $(u_{n_k^{(l+1)}}(x_{l+1}))$ converges. As $(n_k^{(l+1)}) \subseteq \ldots \subseteq (n_k^{(1)})$ we also have that $(u_{n_k^{(k)}}(x_{l+1})), \ldots, (u_{n_k^{(k)}}(x_1)), (u_{n_k^{(k)}}(x_0))$ converge. Hence, $(u_{n_k^{(k)}}(y))$ converges for every $y \in X$. Define

$$u(y) = \lim_{k\to\infty} u_{n_k^{(k)}}(y).$$

Clearly u is positive (as $u_n \geq 0$ and $u(x_0) = u_k(x_0) = 1$).

Assume without loss of generality $(u_{n_k}) = (u_n)$. Now, for any $x \in X$ we have for n large enough, (i.e., $x \in X_n$) that $(\mathcal{L} - \lambda_n)u_n(x) \geq 0$. Hence,

$$\frac{1}{m(x)} \sum_{y\in X} b(x,y)u_n(y) \leq \Big(\frac{1}{m(x)} \sum_{y\in X} b(x,y) + c(x)\Big)u_n(x) - \lambda_n u_n(x).$$

Since $u_n(x) \to u(x)$ and $\lambda_n \to \lambda$ the right hand side converges and, thus, the left side stays bounded. (However, it is not clear that in the limit it is equal to $\frac{1}{m(x)} \sum_y b(x,y)u(x)$.) Nevertheless, by Fatou's lemma

$$\begin{aligned}
\frac{1}{m(x)} \sum_{y\in X} b(x,y)u(y) &\leq \liminf_{n\to\infty} \frac{1}{m(x)} \sum_{y\in X} b(x,y)u_n(y) \\
&\leq \liminf_{n\to\infty} \Big(\Big(\frac{1}{m(x)} \sum_{y\in X} b(x,y) + c(x)\Big)u_n(x) - \lambda_n u_n(x)\Big) \\
&= \Big(\frac{1}{m(x)} \sum_{y\in X} b(x,y) + c(x)\Big)u(x) - \lambda u(x).
\end{aligned}$$

Hence, $u \in \mathcal{F}$ and

$$0 \leq \liminf_{n\to\infty} (\mathcal{L} - \lambda_n)u_n(x) \leq (\mathcal{L} - \lambda)u(x).$$

Thus, u is a positive super-solution and by Harnack inequality it is strictly positive. If the graph is locally finite and u_n are solutions to λ_n on X_n, then the inequality is an equality above and since the sum on the right hand side is finite the limit and the sum interchange. Thus, the second statement follows. □

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5.3 A ground state transform

Let $u > 0$. Define the form Q_u on $C_c(X)$ by

$$Q_u(f) = \frac{1}{2} \sum_{x,y\in X} b(x,y)u(x)u(y)\Big(\frac{f(x)}{u(x)} - \frac{f(y)}{u(y)}\Big)^2.$$

Letting $b_u(x,y) = b(x,y)u(x)u(y)$, we see that $Q_u = Q_{b_u,0} \circ u^{-1}$ on $C_c(X)$. Since $u^{-1}C_c(X) = C_c(X)$, we get that $Q_u(f) < \infty$ by Lemma 4. Moreover, it is obvious that for all $f \in C_c(X)$

$$Q_u(f) \geq 0.$$

This form stands in a close relation to Q if $u > 0$ is a (super)-solution as the next lemma shows.

Lemma 13. *Let $u > 0$ be a solution to λ. Then,*

$$Q(f) = Q_u(f) + \lambda \|f\|, \quad u \in C_c(X).$$

If $u > 0$ is a super-solution then $Q(f) \geq Q_u(f) + \lambda\|f\|$.

Remark In the continuum analogue the formal calculation is as follows

$$\begin{aligned}
\lambda \int f^2 &= \int (\lambda u)\frac{f^2}{u} = \int (-\Delta u)\frac{f^2}{u} = \int \nabla \frac{f^2}{u} \nabla u = \int \frac{1}{u^2}(2uf\nabla f - f^2 \nabla u)\nabla u \\
&= \int \frac{2f}{u}\nabla f \nabla u - \frac{f^2}{u^2}(\nabla u)^2 - (\nabla f)^2 + (\nabla f)^2 \\
&= \int -u^2\Big(\frac{u\nabla f - f\nabla u}{u^2}\Big)^2 + \int (\nabla f)^2 = -\int u^2 (\nabla \frac{f}{u})^2 + \int (\nabla f)^2
\end{aligned}$$

Proof. Let $f \in C_c(X)$ and let u be a strictly positive solution to λ. We employ $\mathcal{L}u = \lambda u$ and Green's formula, Lemma 9, (since $u \in \mathcal{F}$ and $f/u \in C_c(X)$)

$$\lambda \|f\|^2 = \sum_X \lambda u \frac{f^2}{u} m = \sum_X (\mathcal{L}u)\frac{f^2}{u} m = \mathcal{Q}(u, f^2/u).$$

Moreover,

$$\begin{aligned}
&(u(x) - u(y))\Big(\frac{f^2}{u}(x) - \frac{f^2}{u}(y)\Big) \\
&= f^2(x) + f^2(y) - 2f(x)f(y) - u(x)u(y)\frac{f^2}{u^2}(x) - u(x)u(y)\frac{f^2}{u^2}(y) + 2u(x)u(y)\frac{f(x)f(y)}{u(x)u(y)} \\
&= (f(x) - f(y))^2 - u(x)u(y)\Big(\frac{f}{u}(x) - \frac{f}{u}(y)\Big)^2
\end{aligned}$$

Multiplying both terms on the right hand side by $b(x,y)$ and summing over $x, y \in X$ we get by the calculations above the statement.

If u is only a super-solution, then we get an inequality instead of the first equality. The rest of the proof works analogously. $\square$

Remark. Note that $c(x) \geq 0$ is not essential for validity of the lemma.

We can close the form Q_u in the space $\ell^2(X, u^2 m)$ to obtain an operator L_u which is unitarily equivalent to L via the unitary operator

$$U : \ell^2(X, u^2 m) \to \ell^2(X, m), \qquad f \mapsto uf$$

5.4 Proof of the theorem

Proof of Theorem 15. Assume the graph is connected.

(iii)⇒(ii) This is clear.

(ii)⇒(i): Let u be a positive super-solution to $\lambda \in \mathbb{R}$. By Harnack inequality it is strictly positive. By the ground state transform, there a form $Q_u \geq 0$ such that

$$Q(f) \geq Q_u(f) + \lambda \|f\|^2 \geq \lambda \|f\|^2$$

for all $f \in C_c(X)$. By Corollary 3 we get

$$\lambda_0 = \inf_{f \in C_c(X),\, \|f\|=1} Q(f) \geq \lambda.$$

Assume the graph is infinite and locally finite.

(i)⇒(iii) Let $\lambda \leq \lambda_0$. Choose $\lambda_n < \lambda_0$ such that $\lambda_n \to \lambda$, $n \in \mathbb{N}$, (i.e., if $\lambda < \lambda_0$, then choose $\lambda_n = \lambda$), enumerate the vertices $X = \{x_l\}_{l \geq 0}$ and set $X_n = \{x_0, \ldots, x_{n-1}\}$, $n \geq 0$. Let $g_n = (L - \lambda_n)^{-1} \delta_{x_n}$. The resolvent is positivity preserving by Theorem 11. Thus, the function g_n is positive since δ_x is positive. By Harnack inequality g_n is even strictly positive and, in particular, $g_n(x_0) > 0$. Let

$$u_n = \frac{1}{g_n(x_0)} g_n = \frac{1}{(L - \lambda_n)^{-1} \delta_{x_n}(x_0)} (L - \lambda_n)^{-1} \delta_{x_n}.$$

Then, $u_n(x_0) = 1$ and

$$(L - \lambda_n) u_n = \frac{1}{g_n(x_0)} (L - \lambda_n)(L - \lambda_n)^{-1} \delta_n = \frac{1}{g_n(x_0)} \delta_n \geq 0.$$

Since $L = \mathcal{L}$ on $D(L)$, we have $u_n \in \mathcal{F}$ and, moreover, u_n is a solution to λ_n on $X \setminus \{x_n\} \supseteq X_n = \{x_0, \ldots, x_{n-1}\}$. By Proposition 12 there is a strictly positive solution u to λ.

If the graph is finite or not locally finite let $X_n = X \setminus \{x_0\}$, $n \in \mathbb{N}$ and $\lambda_n \to \lambda$, $n \to \infty$. Again by Proposition 12 the function $u_n = g_n / g_n(x_0)$ yields a positive super-solution u. Hence, (i)⇒(ii). □

The 'original' Allegretto Piepenbrink theorem dealt with the essential spectrum of differential operators. Here, is a corresponding analogue.

Corollary 6. *Let (b, c) be infinite, connected and locally finite.*

- *(a) If there is a finite $K \subseteq X$ and a positive super-solution to $\lambda \in \mathbb{R}$ on $X \setminus K$ then, $\lambda \leq \lambda_0^{\mathrm{ess}}$.*
- *(b) For all $\lambda < \lambda_0^{\mathrm{ess}}$ there is a finite subset $K \subseteq X$ and a positive solution to λ on $X \setminus K$.*

Proof. (a) Assume there is a finite set K and a positive super-solution to $\lambda \in \mathbb{R}$. Let Q_K be the closure of the restriction of Q to $C_c(X \setminus K) \subseteq C_c(X)$ and L_K the corresponding operator. By the Allegretto-Piepenbrink theorem $\lambda_0(L_K) \geq \lambda$. Moreover, $L - L_K$ is a finite dimensional operator and, therefore, compact. Hence,

$$\lambda_0^{\text{ess}}(L) = \lambda_0^{\text{ess}}(L_K) \geq \lambda_0(L_K) \geq \lambda.$$

(b) Let $K_n \subseteq X$ be finite, $K_n \subseteq K_{n+1}$ and $X = \bigcup_n K_n$. Then the operators arising from $Q - Q_{K_n}$ are finite dimensional and thus compact. Moreover, as functions in $D(Q_{K_n})$ are supported on $X \setminus K_n$ every sequence (f_n) with $f_n \in D(Q_{K_n})$ is a weak null-sequence. Hence, $\lambda_0(L_{K_n}) \to \lambda_0^{\text{ess}}(L)$, $n \to \infty$ by Proposition 4. Therefore, if $\lambda < \lambda_0^{\text{ess}}(L)$ then there is $n \geq 0$ such that $\lambda_0(L_{K_n}) > \lambda$. By the Allegretto-Piepenbrink theorem there is a positive solution to λ on every connected component of $X \setminus K_n$. □

Ende 15. Vorlesung

weggelassen außer Bäume und Antibäume

5.5 Application to weakly spherically symmetric graphs

We want to consider now graphs with a particular symmetry.

Let (b, c) be a connected graph over a discrete measure space (X, m). Fix a vertex $x_0 \in X$ and call it the *root*. We call such a graph a *rooted graph*.

Define the spheres S_r and the balls B_r with respect to x_0

$$S_r = \{x \in X \mid d(x_0, x) = r\} \quad \text{and} \quad B_r = \bigcup_{j=0}^{r} S_j, \quad r \neq 0.$$

Picture.

Note that a graph is locally finite iff all S_r are finite (**Exercise 30**). Define the functions $b_+, b_- : X \to [0, \infty)$ for $x \in S_r$

$$b_\pm(x) = \frac{1}{m(x)} \sum_{y \in S_{r\pm1}} b(x, y).$$

We call a function $f : X \to \mathbb{R}$ *spherically symmetric* if $f|_{S_r} = \text{const}$ for all $r \geq 0$. In this case we write $f(r) = f(x)$, $x \in S_r$. A graph $(b, 0)$ over (X, m) is called *weakly spherically symmetric* if b_+ and b_- are spherically symmetric functions.

Before we give examples we state the following lemma.

Lemma 14. *For all $r \geq 0$*

$$b_-(r)m(S_r) = b_+(r-1)m(S_{r-1})$$

and if f is a spherically symmetric function, then $\mathcal{L}f$ is spherically symmetric and

$$\mathcal{L}f(r) = b_+(r)(f(r) - f(r+1)) + b_-(r)(f(r) - f(r-1))$$

Proof. **Exercise 31**. □

Example 1. Rooted regular trees. A connected graph is called a tree if it does not contain a closed path (Picture.). If b takes values in $\{0,1\}$ then a rooted graph with root x_0 is a tree iff $b_- \equiv 1$ on $X \setminus \{x_0\}$ (**Exercise 32**). In this case, a rooted tree is called k*-regular* if $b_+ \equiv k$ for some $k \in \mathbb{N}$.
Let $c \equiv 0$ Taking u to be the spherical symmetric function $u(r) = k^{-\frac{r}{2}}$. Then, $(\mathcal{L} - \lambda)u \geq 0$ for $\lambda \leq k + 1 - 2\sqrt{k}$ and, thus,

$$\lambda_0 \geq k + 1 - 2\sqrt{k}.$$

If we let $c = 1_{\{x_0\}}$ then u is even solution to λ. In this case, we can consider c as a Dirichlet boundary condition at x_0 since the backward edge is 'missing'.

2. Antitrees. A connected rooted graph is called an antitree if every vertex of a sphere is connected to all in the previous and next sphere (or equivalently $b_\pm(x) = \#S_{r\pm1}$ for all $x \in S_r$ - **Exercise 33**). (Picture.)

3. Spherically symmetric graphs. The measure space (X, m) is called *spherically symmetric* if m is spherically symmetric and a graph $(b, 0)$ is called *spherically symmetric* if for any $n \geq 0$ and $x, y \in S_n$ there is a graph automorphism γ (that is a bijection $X \to X$ such that $b(u,w) = b(\gamma(u), \gamma(w))$ for all $u, w \in X$) with $\gamma(x_0) = x_0$ and $\gamma(x) = y$. **Exercise 34**: A spherically symmetric graph over a spherically symmetric measure space is weakly spherically symmetric.

4. Picture.

Define the volume the boundary ∂K of a finite set $K \subseteq X$

$$|\partial K| = \sum_{x \in K} \sum_{y \notin K} b(x,y).$$

Note that for a weakly spherically symmetric graph we have

$$|\partial B_r| = \sum_{x \in S_r} b_+(x) m(x) = b_+(r) m(S_r)$$

We will prove the following theorem.

Theorem 17. *If $(b, 0)$ is a weakly spherically symmetric graph over (X, m). If $\left(\sum_{r=0}^{\infty} \frac{m(B_r)}{|\partial B_r|}\right) < \infty$, then*

$$\left(\sum_{r=0}^{\infty} \frac{m(B_r)}{|\partial B_r|}\right)^{-1} \leq \lambda_0 \quad \textit{and} \quad \sigma_{\text{ess}}(L) = \emptyset.$$

From now on let $c \equiv 0$ and b be a locally finite weakly spherically symmetric graph. The strategy is to construct a positive solution for $1/a$ with $a = \sum_{r=0}^{\infty} \frac{m(B_r)}{\partial B_r}$.

Lemma 15. *(Recursion formula for solutions) Let $(b, 0)$ be a weakly spherically symmetric graph over (X, m) and $\lambda \in \mathbb{R}$. A spherically symmetric function u is a solution to λ if and only if*

$$u(r+1) - u(r) = \frac{-\lambda}{\partial B_r} \sum_{j=0}^{r} u(j) m(S_j) = \frac{-\lambda}{\partial B_r} \sum_{j=0}^{r} \sum_{x \in S_j} u(x) m(x).$$

In particular, u is uniquely determined by the choice of $u(0)$.

Proof. The proof is by induction. For $r = 0$ the equation $(\mathcal{L} - \lambda)u(0) = 0$ reads

$$b_+(0)(u(0) - u(1)) = \lambda u(0),$$

which gives the statement. Assume the recursion formula holds for $r - 1$, $r \geq 1$. Then, $(\mathcal{L} - \lambda)u(r) = 0$ reads

$$b_+(r)(u(r) - u(r+1)) + b_-(r)(u(r) - u(r-1)) - \lambda u(r) = 0.$$

Therefore,

$$\begin{aligned} u(r+1) - u(r) &= \frac{b_-(r)}{b_+(r)}(u(r) - u(r-1)) - \frac{\lambda}{b_+(r)} u(r) \\ &= \frac{b_-(r)}{b_+(r)} \frac{(-\lambda)}{b_+(r-1)m(S_{r-1})} \sum_{j=0}^{r-1} u(j)m(S_j) + \frac{(-\lambda)}{b_+(r)m(S_r)} u(r)m(S_r) \\ &= \frac{-\lambda}{b_+(r)m(S_r)} \sum_{j=0}^{r} u(j)m(S_j) \end{aligned}$$

as $b_+(r-1)m(S_{r-1}) = b_-(r)m(S_r)$. □

Lemma 16. *(Existence of positive solutions) Suppose that $a = \sum_{r=0}^{\infty} \frac{m(B_r)}{\partial B_r} < \infty$. Then, there is a solution u to $\frac{1}{a}$ which satisfies*

$$u(r) \geq 1 - \frac{1}{a} \sum_{j=0}^{r-1} \frac{m(B_j)}{\partial B_j}.$$

In particular, u is strictly positive.

Proof. Let $u(0) = 1$ and let u be given by Lemma 15 for $\lambda = \frac{1}{a}$. We show by induction

- $u(r) < u(r-1)$
- $u(r) \geq 1 - \frac{1}{a} \sum_{j=0}^{r-1} \frac{m(B_j)}{\partial B_j}$, i.e., $u(r) > 0$.

For $r = 0$ we get from the recursion formula

$$u(1) - u(0) = -\frac{1}{a\partial B_0} m(0)v(0) < \infty$$

which gives $u(1) < u(0)$. Furthermore,

$$u(1) = \Big(1 - \frac{m(B_0)}{a\partial B_0}\Big) u(0)$$

which gives the second statement. Now suppose the two statements for $1, \ldots, r > 0$. By the recursion formula

$$u(r+1) - u(r) = \frac{-\lambda}{\partial B_r} \sum_{j=0}^{r} u(j) m(S_j) > 0$$

since $u(j) > 0$ by assumption. Moreover,

$$\begin{aligned} u(r+1) &= u(r) - \frac{1}{a\partial B_r} \sum_{j=0}^{r} u(j) m(S_j) > u(r) - \frac{m(B_r)}{a\partial B_r} u(0) \\ &\leq 1 - \frac{1}{a} \sum_{j=0}^{r-1} \frac{m(B_j)}{\partial B_j} - \frac{m(B_r)}{a\partial B_r} = 1 - \frac{1}{a} \sum_{j=0}^{r} \frac{m(B_j)}{\partial B_j}. \end{aligned}$$

□

Proof of Theorem 17. The statement about λ_0 follows from the lemma above and Theorem 15. Let us turn the second statement. Let $X_R = X \setminus B_R \cap \{x_0\}$ and m_R such that $m_R|_{X_R} = m|_{X_R}$ and $m(x_0) = m(S_R)$. Moreover let b_R agree with b on $X_R \times X_R$, $b_R(x_0, x) = b_R(x, x_0) = b_-(x)$ for $x \in S_{r+1}$ and zero otherwise. Then the operator L_R associated to b_R over (X_R, m_R) is differs from L only by a finite dimensional operator. Thus, $\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(L_R)$. On the other hand since $m_r(x_0)/|\partial\{x_0\}| = m(B_r)/|\partial B_r|$, we have by Theorem 17

$$\lambda_0(L_R) \geq \Big(\sum_{r=R}^{\infty} \frac{m(B_r)}{\partial B_r}\Big)^{-1} \to \infty, \quad r \to \infty$$

as the sum converges. This shows that $\lambda_0^{\mathrm{ess}}(L) = \infty$, i.e., $\sigma_{\mathrm{ess}}(L) = \emptyset$

□

Let us apply this theorem to the Laplacian Δ on $\ell^2(X)$

$$\Delta f(x) = \sum_{y \sim x} (f(x) - f(y)),$$

i.e., the $b(x, y) = 1$ iff $x \sim y$. In this case, $b_+(x)/b_-(x)$ are the number of forward/backward neighbors. (Picture)

5.5.1 Spherically symmetric trees

A weakly spherically symmetric tree is determined by a sequence of natural numbers (k_r) via $k_r = b_+(r)$. Since $\#S_{r+1} = b_+(r)\#S_r$ we have $\#S_r = \prod_{j=0}^{r-1} b_+(j)$ and

$$\begin{aligned}\frac{m(B_r)}{\partial B_r} &= \frac{\#B_r}{b_+(r)\#S_r} = \frac{1+\sum_{i=0}^{r-1}\prod_{j=0}^{i} b_+(j)}{\prod_{j=0}^{r} b_+(j)}\\ &= \frac{1}{b_+(r)}\Big(1 + b_+(r-1)^{-1} + \ldots + (b_+(r-1)\ldots b_+(0))^{-1}\Big)\end{aligned}$$

By the limit comparison test $\sum_{r=1}\frac{m(B_r)}{\partial B_r} < \infty$ iff

$$\sum_{r=1}^{\infty}\frac{1}{b_+(r)} < \infty.$$

Hence, the threshold for the applicability of the criterion is

$$b_+(r) \sim r.$$

Indeed, if $b_+(r) \leq r$ then the sum diverges and if $b_+(r) \leq r^{1+\varepsilon}$ for $\varepsilon > 0$ the sum converges. For the volume growth we get by the Stirling formula

$$|B_r| = \prod_{j=0}^{r} k_j \sim r! \sim \sqrt{2\pi r}e^{r\log r/e}.$$

(**Exercise 35**.) In the case, where $b_+(r) \geq r^{1+\varepsilon}$ we can conclude $\lambda_0(\Delta) > 0$ and $\sigma_{\mathrm{ess}}(\Delta) = \emptyset$. (Although we know by the above that already for k-regular trees $\lambda_0 > 0$ if $k > 2$.)

5.5.2 Antitrees

A weakly spherically symmetric tree is determined by a sequence (s_r) via $s_0 = 1$ and $s_r = b_+(r-1)$, $r \geq 1$. Since $\#S_{r+1} = b_+(r)\#S_r$ we have $\#S_r = \prod_{j=0}^{r-1} b_+(j)$ and

$$\frac{m(B_r)}{\partial B_r} = \frac{1+\sum_{j=0}^{r-1} b_+(j)}{b_+(r)b_+(r-1)}$$

Hence, the threshold for the applicability of the criterion is

$$s_r \sim r^2.$$

Indeed, if $s_r \leq r^2$ the sum diverges and if $s_r^{2+\varepsilon}$ the sum converges. For the volume this threshold is then

$$|B_r| \sim r^3.$$

(**Exercise 36**.) In the case where $s_r \geq r^{2+\varepsilon}$ we conclude $\lambda_0(\Delta) > 0$ and $\sigma_{\mathrm{ess}}(\Delta) = \emptyset$.

For $s_{r-1} = r^2$, $r \geq 1$ we check that $u : x \mapsto 1/(d(x_0, x) + 1)$ is a solution to $\lambda = 2$, i.e.,

$$(\mathcal{L} - 2)u(0) = \sum_{y \in S_1} u(x_0) - u(y) - 2u(x_0) = 4\big(1 - \frac{1}{2}\big) - 2 = 0,$$

and for $x \in S_r$

$$(\mathcal{L} - 2)u(x) = \sum_{y \in S_{r+1}} \big(\frac{1}{r} - \frac{1}{r+1}\big) + \sum_{y \in S_{r-1}} \big(\frac{1}{r} - \frac{1}{r-1}\big) - \frac{2}{r} = \frac{r+1}{r} - \frac{r-1}{r} - \frac{2}{r} = 0$$

Thus $\lambda_0 \geq 2$.

Hence, if $s_r = r^\beta$ with $\beta \geq 2$, then $\lambda_0(\Delta) > 0$.

Next we prove a result relating subsexponentially growing solutions to the spectrum, in the spirit that if u is a solution to λ with $e^{-\alpha|\cdot|}u \in \ell^2$ for all $\alpha > 0$ then $\lambda \in \sigma(L)$. In particular, if the graph is polynomially growing, then for $u \equiv 1$ we have $\mathcal{L}u = 0$ and u is subexponentially bounded. This would imply $0 \in \sigma(L)$.

However, the example above shows that there are polynomially growing graphs with positive bottom of the spectrum. This shows that the natural graph metric is not suitable to show such a result. Therefore, we explore some other metrics on graphs.

Chapter 6

Metrics on graphs

We first introduce a metric different to the natural graph metric which is more suitable for our purposes. In the following we will isolate crucial properties of this metric.

Let $\mathrm{Deg}_0 = \mathrm{Deg} - c/m$, i.e.,

$$\mathrm{Deg}_0(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y).$$

We define the map

$$\rho(x, y) = \inf_{x = x_0 \sim \ldots \sim x_n = y} \sum_{i=0}^{n-1} \min\{\mathrm{Deg}_0(x_i)^{-\frac{1}{2}}, \mathrm{Deg}_0(x_{i+1})^{-\frac{1}{2}}\}.$$

which is a pseudo-metric, i.e., $\rho : X \times X \to [0, \infty]$ is symmetric and satisfies the triangle inequality (**Exercise 37**). Moreover let

$$\rho_1 = \rho \wedge 1.$$

Example Let $m = \deg$ be the vertex degree (i.e, $\deg(x) = \#\{y \sim x\}$) and $b : X \times X \to \{0, 1\}$ which is associated to $\widetilde{\Delta}$. In this case $\mathrm{Deg} \equiv 1$ and therefore ρ equals the natural graph distance d.

In general the topology induced by ρ does not agree with the discrete topology.

Example of a non discrete space. Let $X = \mathbb{N}_0$ and $m \equiv 1$. Let b be symmetric and $b(0, 2n) = 1/2^{2n}$, $b(2n{-}1, 2n) = 2^{2n}$ and $b(n, m) = 0$ otherwise. Then, $\rho(0, 2n) = 2^{-n}$. Hence, in the topology induced by ρ every neighborhood of 0 is infinite. In particular, $\{0\}$ is no open set and thus this topology is not the discrete topology.

Moreover, ρ is not necessarily a metric which implies that that (X, ρ) is not necessarily a Hausdorff space. Moreover, (X, ρ) is not necessarily locally compact.

Example of a non-Hausdorff space. Let $X = \mathbb{N}_0 \cup \{\infty\}$, $m \equiv 1$, $c \equiv 0$ and b symmetric such that $b(0, 2n) = b(\infty, 2n) = 2^{-n}$, $b(2n, 2n - 1) = 2^{2n}$ and $b(n, m) = 0$

otherwise. Then b satisfies (b1), (b2), (b3). Moreover, $\mathrm{Deg}(0) = \mathrm{Deg}(\infty) = 1$ and $\mathrm{Deg}(2n) \geq 2^{2n}$. Thus, $d(0, \infty) = \inf_n \mathrm{Deg}(2n)^{-\frac{1}{2}} \geq 2^{-n}$ and hence, 0. Thus, (X, ρ) is not a Hausdorff space.

Example of a non-locally compact space. Let $X = \mathbb{N}_0^2$, $m \equiv 1$, $c \equiv 0$ and let b be symmetric and $b((0,0),(m,0)) = 2^{-m-1}$ and $b((m,n),(m,n+1)) = 2^{2(m+n)}/3$, $m, n \geq 0$ and $b \equiv 0$ otherwise. Then, b satisfies (b1), (b2) and (b3). One can think of the graph as a star graph, where the rays are copies of $\mathbb{N}$. Then, $\mathrm{Deg}(0,0) = 1$ and $\mathrm{Deg}(m,n) = 2^{2(m+n)}$. Hence, $\delta((m,n),(m,n+1)) = 2^{-(m+n)}$ and $\delta((0,0),(m,n)) = 2^{-m} \sum_{k=1}^{n} 2^{-k}$, i.e., $2^{-m-1} \leq \delta((0,0),(m,n)) \leq 2^{-m}$. Let $B_\varepsilon(0,0)$ be a ball about $(0,0)$. Choose $\{U_{\varepsilon/2}(0,0)\} \cup \{U_{2(-m-n-1)}(m,n) \mid m,n \geq 0\}$ as an open covering. However, it is impossible to choose an finite subcovering: Let M be the smallest number such that $X_M := \{(M,n) \mid n \geq 0\} \subseteq B_\varepsilon(0,0)$. Then, $X_M \cap B_\varepsilon(0,0) \setminus B_{\varepsilon/2}(0,0)$ is infinite and since $U_{2(-M-n-1)}(M,n)$ are disjoint for $n \geq 0$ we cannot choose a finite subcovering from it.

However, the pathological behavior mentioned above does not occur in the locally finite case.

Furthermore, ρ has various nice properties which we will study separately. In particular, ρ is an intrinsic path (pseudo)-metric with finite jump size.

6.1 Path metrics and a Hopf-Rinow theorem

Let X be a countable set. For a function $\sigma : X \times X \to [0, \infty)$ we let a path of length n be a sequence $(x_0, \ldots, x_n)$ of pairwise distinct elements in X such that $\sigma(x_i, x_{i+1}) > 0$.

If (b, c) is a graph over X then we assume $b(x,y) > 0$ iff $\sigma(x,y) > 0$. However, we do not need a graph for the considerations in this section.

We define the path (pseudo)-metric $\delta = \delta_\sigma : X \to X \to [0, \infty)$ with respect to σ by

$$\delta(x,y) = \inf_{x = x_0 \sim \ldots \sim x_n = y} \sum_{i=0}^{n-1} \sigma(x_i, x_{i+1}).$$

We call δ a *path (pseudo-)metric* and (X, δ) a *path metric space.*

We call a sequence (x_n) *convergent* if there is an $x \in X$ such that $\delta(x, x_n) \to 0$. Note that since (X, δ) is not necessarily a Hausdorff space limits need not to be unique (compare example in the previous section.) Moreover, we say a sequence (x_n) is a *Cauchy sequence* if for all $\varepsilon > 0$ we have $\delta(x_n, x_m) < \varepsilon$ for large m, n. We call (X, δ) *locally finite* if $\#\{y \in X \mid \sigma(x,y) > 0\} < \infty$ for all $x \in X$.

Example 1. The natural graph metric d.
2. For a graph (b, c) over X let $\sigma(x,y) = b(x,y)^{-1}$ if $x \sim y$ and $\sigma(x,y) = 0$ otherwise.
3. The (pseudo)-metrics ρ and ρ_1 defined in the previous section.

Lemma 17. *Let (X, δ) be locally finite.*

(a) δ is a metric, i.e., (X, δ) is Hausdorff.

(b) (X, δ) is locally compact.

(c) A set is compact in (X, δ) if and only if it is finite.

(d) If a sequence of vertices converges in the metric space (X, δ), then it is eventually constant.

Proof. By the local finiteness for each $x \in X$ there is $r = r_x > 0$ such that $\delta(x, y) \geq r$ for all $y \neq x$ and in particular for all y with $\sigma(x, y) > 0$. Hence, for every vertex x the set $\{x\}$ is open (as $\{x\} = B_{r/2}(x)$). Thus, (a) follows immediately. For (b), as $\{x\}$ is open and compact, every vertex has a compact neighborhood and the space is locally compact. For (c), it is clear that finite sets are compact. On the other hand, let $K \subseteq X$ be compact. Choose an open covering by the open sets $\{x\}$, $x \in K$. Thus, K must be finite. For (d), let (x_n) be a sequence converging to x and let N be such that $\delta(x_n, x) < \varepsilon := r_x$ for $n \geq N$. Then, $(x_n)_{n \geq N}$ is constant. □

The length $l(x) \in [0, \infty]$ of a path $x = (x_n)$ (finite or infinite) is defined as

$$l(x) = \sum_{i \geq 1} \sigma(x_i, x_{i+1}).$$

A path (x_n) is called a *geodesic* if $\delta(x_0, x_k) = l(x_0, \ldots, x_n)$ for all $k \geq 0$. A path metric space (X, δ) is said to be *geodesically complete* if all infinite geodesics have infinite length.

Theorem 18. *(Hopf-Rinow type theorem) Let (X, δ) be a locally finite path metric space. Then, following are equivalent:*

(i) (X, δ) is metrically complete.

(ii) (X, δ) is geodesically complete.

(iii) Every distance ball is finite.

(iv) Every bounded and closed set is compact.

In particular, for all $x, y \in X$ there is a path $x_0, \ldots, x_n$ connecting x and y such that $\delta(x, y) = l(x_0, \ldots, x_n)$.

In the case where a path metric space (X, δ) satisfies one of the (equivalent) properties above we call the path metric space (X, δ) *complete*.

Remark (a) The path metric space (X, δ) is complete iff $(X, \delta \wedge s)$ is complete for all $s > 0$. (**Exercise 38**)

(b) The direction ((iii)⇒)(iv)⇒(i) is true for general metric spaces.

The critical direction is (iii)⇒(ii) which is proven by the following lemma. Ende 16. Vorlesung

Lemma 18. *Let (X, δ) be a locally finite path metric space. If there is an infinite distance ball, then there exists an infinite geodesic of bounded length.*

Proof. Let $o \in X$ be the center of the infinite ball B of radius r and let d be the natural graph distance. Let P_n, $n \geq 0$, be the set of finite paths $(x_0, \ldots, x_N)$ such that $x_0 = o$, $d(x_N, o) = n$ and $d(x_k, o) \leq n$ for $k = 0, \ldots, n$.
Claim: $\Gamma_n = \{\gamma \in P_n \mid \gamma \text{ geodesic}, l(\gamma) \leq r\} \neq \emptyset$ for all $n \geq 0$.
Proof of the claim: The set P_n is finite by local finiteness of the graph and thus contains a minimal element $\gamma = (x_0, \ldots, x_N)$ with respect to the length l, i.e. for all $\gamma' \in P_n$ we have $l(\gamma') \geq l(\gamma)$. Indeed, γ is a geodesic: For every path $(x_0', \ldots, x_M')$ with $x_0' = o$ and $x_{M'} = x_N$, we let $m \in \{n, \ldots, M\}$ be such that $(x_0', \ldots, x_m') \in P_n$. By the minimality of γ we infer

$$l(x_0', \ldots, x_M') \geq l(x_0', \ldots, x_m') \geq l(\gamma).$$

It follows that γ is a geodesic. Clearly, $l(\gamma) \leq r$, as otherwise $B \subseteq \{y \in X \mid d(y, o) \leq n-1\}$ which would imply finiteness of B by local finiteness of the path space. Thus, $\gamma \in \Gamma_n$ which proves the claim.

We inductively construct an infinite geodesic (x_k) with bounded length: We set $x_0 = o$. Since $\Gamma_n \neq \emptyset$, there is a geodesic in Γ_n for every $n \geq 0$ such that x_0 is a subgeodesic. Suppose we have constructed a geodesic $(x_1, \ldots, x_k)$ such that for all $n \geq k$ there is a geodesic in Γ_n that has $(x_1, \ldots, x_k)$ as subgeodesic. By local finiteness x_k has finitely many neighbors. Thus, there must be a neighbor x_{k+1} of x_k such that for infinitely many n the path $(x_0, \ldots, x_k, x_{k+1})$ is a subpath of a geodesic in Γ_n. However, a subpath of geodesic is a geodesic. Thus, there is an infinite geodesic $\gamma = (x_n)_{n \geq 0}$ with $l(\gamma) = \lim_{n \to \infty} l(x_0, \ldots, x_n) \leq r$ as $(x_0, \ldots, x_n) \in \Gamma_n$ for all $n \geq 0$. □

Proof of Theorem 18. (i)⇒(ii): If there is a bounded geodesic, then it is a Cauchy sequence. Since a geodesic is a path it is not eventually constant, so it does not converge by Lemma 17 (d). Hence, (X, δ) is not metrically complete.
(ii)⇒(iii): Suppose that there is a distance ball that is infinite. By Lemma 18 there is a bounded infinite geodesic. Thus, (X, δ) is not geodesically complete.
(iii)⇒(iv) follows from Lemma 17 (c). (iv)⇒(i): If every bounded and closed set is compact, then every closed distance ball is compact. Then, by Lemma 17 (c) every distance ball is finite and it follows that (X, δ) is metrically complete (since Cauchy sequences are bounded). □

6.2 Intrinsic metrics

Let (b, c) be a graph over a discrete measure space (X, m).

For $f \in \mathcal{C}(X)$, we define $|d_b f|^2 : X \to [0, \infty]$

$$|d_b f|^2(x) = |d_b f \cdot d_b f(x)| = \sum_{y \in X} b(x, y)(f(x) - f(y))^2.$$

and the functions for which the gradient is finite by

$$\mathcal{D}^*_{loc} = \{f \in C(X) \mid |d_b f|^2(x) < \infty \text{for all } x \in X\}.$$

Lemma 19. *(a)* $D(Q) \subseteq \mathcal{D}^*_{loc}$.

(b) $\mathcal{F} \cap \mathcal{D}^*_{loc} = \mathcal{F}_{\in} = \{f \in C(X) \mid f^2 \in \mathcal{F}\}$*. In particular, if* $f \in \mathcal{F}_2$ *then* $fg \in \mathcal{F}_2$ *for* $g \in \ell^\infty$*.*

Proof. (a) For $f \in D(Q)$ we have for all $x \in X$

$$|d_b f|^2(x) \le \sum_X |d_b f|^2 = 2Q(f) < \infty.$$

(b) Let $f \in F_2$. Then, $f \in \mathcal{F}$ (as $\sum_y b(x,y) < \infty$). In this case

$$|d_b f|^2(x) = f(x)^2 \underbrace{\sum_{y\in X} b(x,y)}_{<\infty,\,(\mathrm{b3})} - 2f(x) \underbrace{\sum_{y\in X} b(x,y)f(y)}_{<\infty,\, f\in\mathcal{F}} + \underbrace{\sum_{y\in X} b(x,y)f(y)^2}_{<\infty,\, f\in\mathcal{F}_2} < \infty,$$

so $f \in \mathcal{D}^*_{loc}$. On the other hand, let $f \in \mathcal{F} \cap \mathcal{D}^*_{loc}$. Then,

$$\sum_{y\in X} b(x,y)f(y)^2 = \underbrace{|d_b f|^2(x)}_{<\infty,\, f\in\mathcal{D}^*_{loc}} + 2f(x) \underbrace{\sum_{y\in X} b(x,y)f(y)}_{<\infty,\, f\in\mathcal{F}} - f(x)^2 \underbrace{\sum_{y\in X} b(x,y)}_{<\infty,\,(\mathrm{b3})} < \infty,$$

and, thus, $f \in \mathcal{F}_2$. Clearly, $\sum_{y\in X} b(x,y)f(y)^2 g(y)^2 \le \|g\|_\infty^2 \sum_{y\in X} b(x,y)f(y)^2 < \infty$ for $g \in \ell^\infty(X)$. □

Let δ be a pseudo metric on X. For $A \subseteq X$, we define

$$\delta_A(x) = \inf_{y\in A} \delta(x,y), \quad x \in X.$$

Moreover, $\delta \wedge a = \min\{\delta, a\}$, $a \ge 0$ is a pseudo metric and one has

$$(\delta \wedge a)_A = \delta_A \wedge a$$

and for all $A \subseteq X$ and $a \ge 0$

$$|\delta_A(x) \wedge a - \delta_A(y) \wedge a| \le \delta(x,y), \quad x, y \in X. \tag{6.1}$$

Lemma 20. *Let* δ *be a pseudo metric. Then the following are equivalent*

(i) $|d_b(\delta_A \wedge a)|^2 \le m$ *for all* $A \subseteq X$ *and* $a \ge 0$*.*

(ii) $\sum_{y\in X} b(x,y)\delta(x,y)^2 \le m(x)$*.*

In particular, in this case $\delta_K \in \mathcal{D}^*_{loc}$ *for all finite* $K \subseteq X$*.*

Proof. Suppose (i). Let $x \in X$ and $A = \{x\}$. We obtain

$$\begin{aligned}\sum_{y\in X} b(x,y)(\delta(x,y)\wedge a)^2 &= \sum_{y\in X} b(x,y)(\delta(x,x)-\delta(x,y))^2\\ &= \sum_{y\in X} b(x,y)(\delta_A(x)\wedge a - \delta_A(y)\wedge a)^2\\ &= |d_b\delta_A\wedge a|^2(x) \le m(x).\end{aligned}$$

By Lebesgue's Theorem

$$|d_b\delta_A|^2 = \lim_{a\to\infty} |d_b(\delta_A\wedge a)^2| \le m$$

which is (ii). On the other hand, if (ii) we have by (6.1)

$$|d_b(\delta_A\wedge a)|^2(x) = \sum_{y\in X} b(x,y)(\delta_A(x)\wedge a - \delta_A(y)\wedge a)^2 \le \sum_{y\in X} b(x,y)\delta(x,y)^2 \le m(x).$$

□

A pseudo metric $\rho : X\times X \to [0,\infty]$ that satisfies (i) or (ii) is called an *intrinsic metric.*

Exercise 39 If δ is an intrinsic metric, so is $\delta\wedge s$ for all $s > 0$.

Example The pseudo metric ρ from the previous section is an intrinsic metric. In particular,

$$\sum_{y\in X} b(x,y)\rho(x,y)^2 \le \sum_{y\in X} b(x,y)\min\{\frac{1}{\mathrm{Deg}(x)},\frac{1}{\mathrm{Deg}(y)}\} \le \mathrm{Deg}(x)\sum_{y\in X} b(x,y) \le m(x)$$

Suppose L associated to (b,c) is a bounded operator which implies $\mathrm{Deg}\equiv C$ by Theorem 8. Then, $d/C \le \rho$, where d the natural graph metric. Thus, δ/C is an intrinsic metric. In particular, if $\mathrm{Deg}\equiv 1$ (as in the case of $\widetilde{\Delta}$), then $d=\rho$ is an intrinsic metric.

Let the Lipshitz continuous functions be given as

$$\mathrm{Lip} = \{f\in C(X) \mid \text{there exists } C \text{ such that } f(x)-f(y) \le C\delta(x,y) \text{ for all } x,y\in X\}.$$

For a function $f\in \mathrm{Lip}$ the constant C is called the Lipshitz constant.

Lemma 21. *Let δ be an intrinsic metric.*

(a) $|d_b\eta|^2 \le C^2 m$ *for all* $\eta\in \mathrm{Lip}$ *with Lipshitz constant* C*. In particular,* $\mathrm{Lip}\subseteq \mathcal{D}^*_{loc}$.

(b) $\eta D(Q)\subseteq D(Q)$ *and* $\eta\mathcal{D}^*_{loc}\subseteq \mathcal{D}^*_{loc}$ *for all* $\eta\in \mathrm{Lip}\cap \ell^\infty(X)$.

Proof. (a) Let $\eta \in \mathrm{Lip}$. Then,

$$|d_b\eta|^2(x) = \sum_{y\in X} b(x,y)(\eta(x)-\eta(y))^2 \le C^2 \sum_{y\in X} b(x,y)\delta(x,y)^2 \le C^2 m(x),$$

since δ is intrinsic. It directly follows that $\eta \in \mathcal{D}^*_{loc}$.
(b) For $\eta, g \in C(X)$ we have

$$(\eta g)(x) - (\eta g)(y) = g(x)(\eta(x)-\eta(y)) - \eta(y)(g(x)-g(y))$$

and for $\eta \in \mathrm{Lip} \cap \ell^\infty(X)$ we have

$$\begin{aligned}|(\eta g)(x) - (\eta g)(y)|^2 &= 2|g(y)|^2|\eta(x)-\eta(y)|^2 + 2|\eta(x)|^2|g(x)-g(y)|^2 \\ &\le 2|g(y)|^2\delta(x,y)^2 + 2\|\eta\|_\infty^2|g(x)-g(y)|^2.\end{aligned}$$

For $g \in \mathcal{D}^*_{loc}$ notice that

$$\begin{aligned}|d_b\eta g|^2(x) &\le 2g(x)^2 \sum_{y\in X} b(x,y)\delta(x,y)^2 + 2\|\eta\|_\infty^2 \sum_{y\in X} b(x,y)(g(x)-g(y))^2 \\ &\le 2g(x)^2 m(x) + 2\|\eta\|_\infty^2 |d_b g|^2(x)\end{aligned}$$

which implies $\eta g \in \mathcal{D}^*_{loc}$.

We are left to show that $\eta g \in D(Q)$ for $g \in D(Q)$ and $\eta \in \mathrm{Lip} \cap \ell^\infty(X)$. Let first $g \in C_c(X)$ which implies $\eta g \in C_c(X)$ and thus $\eta g \in D(Q)$. Summing the inequality above over x and multiplying by $\frac{1}{2}$, we obtain

$$Q(\eta g) \le \|g\|^2 + \|\eta\|_\infty^2 Q(g)$$

Moreover, we have that $\|\eta g\| \le \|\eta\|_\infty \|g\|$. This implies that if (f_n) is a $\|\cdot\|_Q = \sqrt{Q(\cdot) + \|\cdot\|^2}$ Cauchy sequence in $C_c(X)$, then so is (ηf_n) (let $g = f_n - f_m$ in the estimates above). Now, as $D(Q)$ is the closure of $C_c(X)$ with respect to $\|\cdot\|_Q$ we have that $\eta f \in D(Q)$ for $f \in D(Q)$ (since ηf_n converge to ηf in $\ell^2(X,m)$). This finishes the proof of (b). □

Ende 16. Vorlesung

The following functions will play an important role. For $U \subseteq X$ and $r > 0$ let

$$\eta_{U,r} := \left(1 - \frac{\delta_U}{r}\right) \vee 0.$$

(Picture)

Lemma 22. *For any $U \subseteq X$ and $r > 0$ we have $\eta_{U,r} \in \mathrm{Lip}$ with Lipshitz constant $1/a^2$. In particular, $\eta_{U,r} \in \mathcal{D}^*_{loc}$ and $|d_b\eta_{U,r}|^2 \le \frac{1}{r^2}m$. Moreover, if the ball $B_r(U) = \{y \in X \mid \delta_U(y) \le r\}$ is finite, then $\eta_{U,r} \in C_c(X) \subseteq D(Q)$.*

Proof. Let $\eta = \eta_{U,r}$. We estimate using (6.1)

$$
\begin{aligned}
(\eta(x) - \eta(y))^2 &= \Big(\Big(\Big(1 - \frac{\delta_U}{a}\Big) \vee 0\Big)(x) - \Big(\Big(1 - \frac{\delta_U}{r}\Big) \vee 0\Big)(y)\Big)^2 \\
&\leq \Big(\frac{\delta_U(x)}{r} - \frac{\delta_U(y)}{r}\Big)^2 \\
&\leq \frac{1}{r^2}\delta(x,y)^2.
\end{aligned}
$$

This implies $\eta \in \mathrm{Lip}$ with Lipshitz constant $1/r^2$. The 'in particular' follows from Lemma 21. The last statement is obvious as $\operatorname{supp} \eta = B_r(U)$. □

We call

$$s := \inf\{t \geq 0 \mid \delta(x,y) \leq t \text{ for } x \sim y\}$$

the *jump size* of the pseudo metric δ. In particular, if δ is an intrinsic metric for a graph (b,c) over (X,m) we call s also the jump size of Q.

Remark Given an intrinsic metric δ and $s > 0$. Then $\delta \wedge s$ is an intrinsic metric and has jump size s. In particular, $\rho_1 = \rho \wedge 1$ is an intrinsic metric with jump size.

Lemma 23. *Let s be the jump size of Q, let $U \subseteq X$ and $r > 0$. Then, for $A_{r+s}(U) = B_{r+s}(U) \cap B_{r+s}(X \setminus U)$*

$$|d_b \eta_{U,r}|^2 \leq \frac{1}{r^2} 1_{A_{r+s}(U)} m.$$

Proof. Let $\eta = \eta_{U,r}$. For $x \in X \setminus B_{r+s}(U)$, we have $b(x,y)(\eta(x)-\eta(y)) = b(x,y)\eta(y) = 0$ since $\eta(y) = 0$ for $y \in X \setminus B_r(U)$ and $b(x,y) = 0$ for $y \in B_r(U)$ (as $\delta(x,y) \geq s$ in this case). On the other hand if $x \in X \setminus B_{r+s}(X \setminus U)$ we have $b(x,y)(\eta(x) - \eta(y)) = b(x,y)(1 - \eta(y)) = 0$ since $\eta(y) = 1$ for $y \in U$ and $b(x,y) = 0$ for $y \in X \setminus U$ (as $\delta(x,y) \geq s + r$ in this case). Thus $b(x,y)(\eta(x) - \eta(y)) = 0$ for $x \notin A_{U,r}$. Hence, by Lemma 22 we have

$$|d_b \eta|^2 = 1_{A_{r+s}(U)} |d_b \eta|^2 \leq \frac{1}{r^2} 1_{A_{r+s}(U)} m.$$

□

Chapter 7

Subexponentially bounded solutions

Let (b, c) be a graph over (X, m). Morever, let Q be the corresponding form and L the corresponding operator.

Let δ be an intrinsic metric, i.e.,

$$\sum_{y \in X} b(x,y)\delta(x,y)^2 \leq m(x).$$

Let s be the jump size, (i.e., $b(x,y) = 0$ for $\delta(x,y) > s$) and assume $s < \infty$.

We call a function $f \in C(X)$ *subexponentially bounded* if for some x_0 and all $\alpha > 0$

$$e^{-\alpha\delta(x_0,\cdot)} f \in \ell^2(X, m).$$

Example and Exercise 40 (a) Let $X = \mathbb{Z}^n$, $m \equiv 1$ and $b : X \times X \to \{0, 1\}$. Then, $d_n = d/2n$ is an intrinsic metric where d is the natural graph metric. A function f is subexponentially bounded iff

$$\limsup_{k\to\infty} \sup_{x \in B_k(0)} \frac{1}{k} \log |f(k)| \leq 0.$$

(b) Let $X = T_k$, $m \equiv 1$ and $b : X \times X \to \{0, 1\}$. Then, $d_k = d/(k+1)$ is an intrinsic metric. A function f is subexponentially bounded iff

$$\limsup_{n\to\infty} \sup_{x \in B_n(x_0)} \frac{1}{n} \log |f(n)| \leq -\log k/2$$

The next theorem says that if there is a subexponentially bounded solution to some λ, then $\lambda \in \sigma(L)$.

Theorem 19. *(Shnol' theorem) Let the graph be locally finite, δ be an intrinsic metric such that the jump size is finite and such that all distance balls are finite. If there is a subexponentially bounded solution $u \in C(X)$ to $\lambda \in \mathbb{R}$ then $\lambda \in \sigma(L)$.*

Remark The assumptions finite jump size and finiteness of distance balls implies locally finiteness of the graph. (Indeed, if s is the jump size and x is a vertex with infinite degree, then the s-ball about x is infinite.)

The following corollary for the intrinsic path metric ρ induced by

$$\rho(x, y) = \min\{\mathrm{Deg}(x)^{-\frac{1}{2}}, \mathrm{Deg}(y)^{-\frac{1}{2}}\} \wedge 1, \quad x \sim y$$

The following theorem follows directly from the Hopf-Rinow theorem.

Corollary 7. *Let (b, c) be locally finite and (X, ρ) be complete with finite jump size. If there is a subexponentially bounded solution $u \in \mathcal{D}^*_{loc} \cap \mathcal{F}$ to $\lambda \in \mathbb{R}$, then $\lambda \in \sigma(L)$.*

7.1 A Caccioppoli inequality

Note that we do not need the assumptions that imply local finiteness for the following inequality.

Theorem 20. *(Caccioppoli inequality) Let $u \in \mathcal{D}^*_{loc} \cap \mathcal{F}$ be a solution to $\lambda \in \mathbb{R}$. Then, there is $C > 0$ such that for all $v \in C_c(X)$*

$$\sum_X v^2 |d_b u|^2 \leq C\Big(\|uv\|^2 + \sum_X u^2 |d_b v|^2\Big)$$

Proof. Assume w.l.o.g. $c \equiv 0$. Since u is a solution to λ, Green's formula and the Leibniz rule yields

$$\lambda \|uv\|^2 = \sum_X (\mathcal{L}u)uv^2 = \mathcal{Q}(u, uv^2) = \sum_X v^2 |d_b u|^2 + \sum_X u(d_b u \cdot d_b v^2).$$

Note that the left hand side is finite since $u \in \mathcal{F}$ and $uv^2 \in C_c(X)$ and the first term on the right hand side is in finite since $u \in \mathcal{D}^*_{loc}$ and $v \in C_c(X)$. Moreover, for the second term

$$\begin{aligned}
\Big|\sum_X u(d_b u \cdot d_b v^2)\Big| &\leq \sum_{x,y \in X} b(x, y)|u(x)(v(x) + v(y))(u(x) - u(y))(v(x) - v(y))| \\
&\leq \frac{1}{\varepsilon} \sum_X u^2 |d_b v|^2 + 4\varepsilon \sum_{x,y \in X} b(x, y)(v(x) + v(y))^2 (u(x) - u(y))^2 \\
&\leq \frac{1}{\varepsilon} \sum_X u^2 |d_b v|^2 + 8\varepsilon \sum_X v^2 |d_b u|^2,
\end{aligned}$$

where we used $2ab \leq a^2/\varepsilon + 4\varepsilon b^2$ for $\varepsilon > 0$. Thus,

$$\begin{aligned}
\sum_X v^2 |d_b u|^2 &= \lambda \|uv\|^2 - \sum_X u(d_b u \cdot d_b v^2) \\
&\leq \lambda \|uv\|^2 + \frac{1}{\varepsilon} \sum_X u^2 |d_b v|^2 + 4\varepsilon \sum_X v^2 |d_b u|^2 \\
&= \frac{1}{1 - 4\varepsilon}\Big(\lambda \|uv\|^2 + \frac{1}{\varepsilon} \sum_X u^2 |d_b v|^2\Big).
\end{aligned}$$

□

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7.2 A Shnol' inequality

Let δ be a pseudo metric. Recall the definitions

$$\delta_U = \inf_{x\in U} \delta(x,\cdot),$$
$$\eta_{U,r} = \Big(1 - \frac{\delta_U}{r}\Big) \vee 0,$$
$$A_r(U) = B_r(U) \cap B_r(X \setminus U),$$

where $U \subseteq X$ and $r \geq 0$.

Lemma 24. *(Shnol' type inequality) Let an intrinsic metric δ be given and let $s > 0$ be the jump size. Let $v \in C_c(X)$ and $u \in \mathcal{D}^*_{loc} \cap \mathcal{F}$ be a solution to $\lambda \in \mathbb{R}$. Then, for $U \subseteq X$ and $r > 0$ such that $B_{r+s}(U)$ is finite there is $C > 0$ such that*

$$|(Q-\lambda)(u\eta^2_{U,r}, v)| \leq C\|v\|_Q \|u1_{A_{r+s}(U)}\|$$

Proof. Let $\eta = \eta_{U,r}$. By assumption $\eta \in C_c(X)$. Using $(\mathcal{Q}-\lambda)(u,w) = \sum((\mathcal{L} - \lambda)u)wm = 0$ for all $w \in C_c(X)$ we get by Leibniz rule

$$\begin{aligned}(Q-\lambda)(u\eta^2, v) &= (\mathcal{Q}-\lambda)(u\eta^2, v) - (\mathcal{Q}-\lambda)(u, \eta^2 v)\\ &= \sum_X u(d_b\eta^2 \cdot d_b v) + \underbrace{\sum_X \eta^2(d_b u \cdot d_b v)}_{<\infty \text{ by CSI}} - \underbrace{\sum_X \eta^2(d_b u \cdot d_b v)}_{<\infty \text{ by CSI}} - \sum_X v(d_b\eta^2 \cdot d_b u)\\ &= 2\sum_X u\eta(d_b\eta \cdot d_b v) - 2\sum_X v\eta(d_b\eta \cdot d_b u 1_{A_{r+s}})\end{aligned}$$

since

$$\begin{aligned}\sum_X v\eta(d_b\eta \cdot d_b u) &= \sum_{x,y\in X} \underbrace{b(x,y)(\eta(x)-\eta(y))}_{=0 \text{ for } x \text{ or } y\in X\setminus A_{r+s}(U)} \eta(x)v(x)(u(x)-u(y))\\ &= \sum_X v\eta(d_b\eta \cdot d_b u 1_{A_{r+s}}).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$\ldots \leq 2Q(v)^{\frac{1}{2}}\Big(\sum_X u^2\eta^2|d_b\eta|^2\Big)^{\frac{1}{2}} + 2\Big(\sum_X \eta^2|d_b u 1_{A_{r+s}(U)}|^2\Big)^{\frac{1}{2}}\Big(\sum_X v^2|d_b\eta|^2\Big)^{\frac{1}{2}}$$

By Lemma 21 we have $\sum_X v^2|d_b\eta|^2 \leq \sum_X v^2 m/r^2 = \|v\|^2/r^2$. By Caccioppoli inequality and $\|\eta\|_\infty \leq 1$

$$\begin{aligned}\ldots &\leq 2Q(v)^{\frac{1}{2}}\Big(\sum_X u^2|d_b\eta|^2\Big)^{\frac{1}{2}} + 2C\Big(\|u1_{A_{r+s}(U)}\|^2 + \sum_X u^2|d_b\eta|^2\Big)^{\frac{1}{2}}\|v\|\\ &\leq C(Q(v)^{\frac{1}{2}} + \|v\|)\|u1_{A_{r+s}(U)}\|\end{aligned}$$

where we used $|d_b\eta|^2 \leq 1_{A_{r+s}(U)}m/r^2$ by Lemma 23 which finishes the proof. □

7.3 A general Shnol' theorem

Theorem 21. *Let δ be an intrinsic metric and let s be the jump size. Suppose there is a sequence (U_n) of subsets such that $B_{r+s}(U_n)$ are finite for some $r > 0$ and*

$$\frac{\|u1_{A_{r+s}(U_n)}\|}{\|u1_{U_n}\|} \to 0$$

for some solution $u \in \mathcal{D}^*_{loc} \cap \mathcal{F}$ to $\lambda \in \mathbb{R}$. Then, $\lambda \in \sigma(L)$.

Proof. Let u be a solution. Since $\eta_{U_n,r} \in C_c(X)$ we have $u_n = u\eta^2_{U_n,r} \in C_c(X) \subseteq D(Q)$. As $\|u_n\| \geq \|u1_{U_n}\|$ we get for all $v \in C_c(X)$ with $\|v\|_Q = 1$

$$\frac{|(Q-\lambda)(u_n, v)|}{\|u_n\|} \leq C\frac{\|u1_{A_{r+s}(U)}\|}{\|u_n\|} \leq C\frac{\|u1_{A_{r+s}(U)}\|}{\|u1_{U_n}\|} \to 0$$

for $n \to \infty$, by the Shnol' inequality, Lemma 24. Thus, $(u_n/\|u_n\|)$ is a form Weyl sequence and the statement follows from Theorem 12. □

Lemma 25. *Let $J : [0,\infty) \to [0,\infty)$ be such that for all $\alpha > 0$ there is $C_\alpha > 0$ such that $J(r) \leq C_\alpha e^{\alpha r}$ for all $r \geq 0$. Then for all $m > 0$ and $\delta > 0$ there exist an unbounded sequence of numbers $r_k > 0$ such that $J(r_k + m) \leq e^\delta J(r_k)$.*

Proof. Assume the contrary. Then there exists an $r_0 \geq 0$ such that $J(r_0) \neq 0$ and $J(r+m) > e^\delta J(r)$ for all $r > r_0$. By induction we get $J(r_0 + nm) > e^{n\delta} J(r_0)$ for $n \geq 1$. For $\alpha < \delta/m$ we get by $J(r) \leq C_\alpha e^{\alpha r}$

$$J(r_0) \leq J(r_0 + mn)e^{-\delta n} \leq C_\alpha e^{\alpha r_0} e^{(\alpha m - \delta)n} \to 0,$$

as $n \to \infty$. This is a contradiction to $J(r_0) \neq 0$. □

Proof of Shnol's theorem. Let $u \in \mathcal{D}^*_{loc}$ be a subexponentially bounded solution to λ and $u_n = 1_{B_n(x_0)}$. Then for all $\alpha > 0$

$$\|u_n\|^2 = \sum_{x \in B_n} |e^{\alpha d(x_0,x)} e^{-\alpha d(x_0,x)} u(x)|^2 m(x) \leq e^{2\alpha n} \|e^{-\alpha d(x_0,\cdot)} u\|^2,$$

which implies that $n \mapsto \|u_n\|^2$ satisfies the assumption of the lemma above. Hence, for all $n \geq 1$ there is a sequence $(r_k^{(n)})$ such that $\|u_{r_k^{(n)}+r+1}\|^2 \leq e^{1/n} \|u_{r_k^{(n)}-r-1}\|^2$, where s is the jump size. Letting $j_n = r_n^{(n)}$ we get

$$\frac{\|u1_{A_{r+s}(B_n)}\|^2}{\|u_n\|^2} = \frac{\|u_{j_n+r+s}\|^2 - \|u1_{j_n-r-s}\|^2}{\|u_n\|^2} \leq (e^{1/n} - 1)\frac{\|u_{j_n-r-s}\|^2}{\|u_n\|^2} \to 0$$

Thus, $\lambda \in \sigma(L)$ follows from Theorem 21 since all distance balls are finite. □

7.4 Applications

Moreover, we get the following special case of Brook's theorem.

Corollary 8. *(Baby Brooks) Let δ be an intrinsic metric such that the jump size is finite and such that all distance balls are finite. If the graph is of subexponential growth, i.e.* $\limsup \frac{1}{r} \log m(B_r(x_0)) \leq 0$ *for some (all)* $x_0 \in X$*, then* $0 \in \sigma(L)$.

Proof. The constant functions are solutions to $\lambda = 0$. Moreover, they are clearly contained in $\mathcal{D}^*_{loc} \cap \mathcal{F}$ and under the assumption of subexponential growth the constant functions are subexponentially bounded. □

Let $b : X \times X \to \{0,1\}$, $c \equiv 0$ and $m \equiv 1$. Then L is the operator Δ with

$$\Delta f(x) = \sum_{y \sim x} (f(x) - f(y))$$

7.4.1 The Euclidian lattice

The Euclidian lattice is the graph with vertex set $\mathbb{Z}^n$ and $b(n,m) = 1$ if and only if $|n - m| = 1$.

The constant function $u \equiv 1$ is harmonic, i.e., for $\lambda = 0$ we have $(\mathcal{L} + \lambda)u = \mathcal{L}u = 0$. Moreover, since $\mathbb{Z}^n$ has polynomial growth, the function $u \equiv 1$ is subexponentially bounded. Thus, $0 \in \sigma(\Delta)$ and since Δ is positive $\lambda_0 = 0$.

In general, for every graph with bounded the vertex degree and subexpontial growth, we have $0 \in \sigma(\Delta)$.

7.4.2 Trees

Let the graph be an unrooted k-regular tree, $k \geq 2$ and $c = 1_{x_0}$. In the section about positive solutions we found that $u : x :\mapsto 1/d(x,x_0)^{k/2}$ satisfies $(\mathcal{L} - \lambda)u = 0$ for $\lambda = k + 1 - 2\sqrt{k}$. Since

$$\frac{1}{n} \log u(n) = -\log k/2,$$

we have $\lambda \in \sigma(L)$. As $\lambda \leq \lambda_0$ by positivity of u we have

$$\lambda_0 = k + 1 - 2\sqrt{k}.$$

7.4.3 Antitrees

An antitree is determined by a sequence of natural numbers (s_n) such that $|S_n| = s_n$ and every vertex in a sphere is connected to all vertices in the previous and next sphere.

In the section about positive solutions we learned that if $s_r \geq r^{2+\varepsilon}$, for $\varepsilon > 0$ we have

$$\lambda_0(\Delta) = 0 \quad \text{and} \quad \sigma_{\text{ess}}(\Delta) = 0.$$

We are interested in the borderline, where $\lambda_0 > 0$.

Recall the path metric ρ given by

$$\rho(x, y) = \inf_{x=x_0\sim\ldots\sim x_n=y} \sum_{i=1}^{n} \min\{\text{Deg}_0(x_i)^{-\frac{1}{2}}, \text{Deg}_0(x_{i+1})^{-\frac{1}{2}}\}$$

for $x, y \in X$ and $\text{Deg}_0(x') = \frac{1}{m(x')} \sum_{y' \in X} b(x', y')$.

Assume the graph is an antitrees with monotone increasing (s_n). Then, for $x \in S_n$

$$\rho(x_0, x) = \sum_{j=1}^{n} \frac{1}{(s_{j-1} + s_{j+1})^{\frac{1}{2}}}.$$

Let $s_n = [n^\beta]$, where $0 < \beta \leq 2$ and $[r]$ is the smallest integer which is larger than r. Then, for $x \in S_n$ and $\beta \in (0, 2)$

$$\rho(x_0, x) = \sum_{j=1}^{n} \frac{1}{([(j-1)^\beta] + [(j+1)^\beta])^{\frac{1}{2}}} \sim \int_1^n j^{-\frac{\beta}{2}} dj \sim n^{1-\frac{\beta}{2}},$$

where $a_n \sim b_n$ means there is $C > 0$ such that $C^{-1} a_n \leq b_n \leq C a_n$. Let B_r, $r \geq 0$ the ball with respect to ρ. Let $\beta \in (0, 2)$. For $n \in \mathbb{N}$ and $r = n^{1-\frac{\beta}{2}}$

$$m(B_r) = |B_r| \sim \sum_{j=1}^{n} s_j = \sum_{j=1}^{n} j^\beta \sim \int_1^n j^\beta dj \sim n^{\beta+1} = r^{2\frac{\beta+1}{2-\beta}}.$$

Thus, there is $C > 0$ such that

$$C^{-1} r^{2\frac{\beta+1}{2-\beta}} \leq m(B_r) \leq C r^{2\frac{\beta+1}{2-\beta}}.$$

Thus $\mu = 0$ and by the corollary 'Baby Brooks', we have that $0 \in \sigma(\Delta)$.

Hence, we conclude for $s_r = r^\beta$ that

$$\begin{aligned} \lambda_0(\Delta) &= 0, \quad \text{for } 0 < \beta < 2 \\ \lambda_0(\Delta) &> 0, \quad \text{for } \beta > 2 \end{aligned}$$

Exercise 41: Check that for $s_r = r^2$ the volume grows exponentially.

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Chapter 8

Volume growth and upper bounds

8.1 Motivation

For Riemannian manifolds we have the following theorem which goes back to Brooks: Let M be a Riemannian manifold and $B_r = B_r(x_0)$ the ball of radius r about $x_0 \in M$ in the Riemannian metric

$$\mu = \limsup_{r\to\infty} \frac{1}{r} \log \mathrm{vol}(B_r)$$

i.e., $\mathrm{vol}(B_r) \sim e^{\mu r}$. For the Laplace Beltrami operator Δ_M, we have

$$\lambda_0^{\mathrm{ess}}(\Delta_M) \leq \frac{\mu^2}{4}.$$

In particular, this implies that if M has polynomial growth then $\lambda_0 = \lambda_0^{\mathrm{ess}} = 0$.

The idea of the proof is as follows: Let $\alpha > \mu/2$ and $g_n|_{B_r} \equiv n$ and $g_r(x) = e^{\alpha(2r-d(x_0,x)}$ for $x \notin B_r$. Then,

$$\int_M |\nabla g_r|^2 = \alpha^2 \int_{M\setminus B_r} |g_r|^2 \leq \alpha^2 \|g_r\|^2.$$

By showing that g_r are in the form domain for $\alpha > \mu/2$ and $g_r/\|g_r\| \to 0$ weakly we get the statement. As we have seen in Section 5.5, there are graphs of polynomial growth (with respect to the natural graph metric) which satisfy $\lambda_0 > 0$ and thus $\lambda_0^{\mathrm{ess}} > 0$. However, for intrinsics metrics we can proof a graph analogue of Brooks theorem.

Let $(b, 0)$ be a connected graph over a discrete measure space (X, m). Let Q be the corresponding quadratic form and L the associated operator. Let δ be an intrinsic metric.

Moreover, let B_r be the distance balls about a fixed vertex $x_0 \in X$. Define

$$\mu := \limsup_{r\to\infty} \frac{1}{r} \log m(B_r).$$

Lemma 26. *If* $\bigcup B_r(x) = X$ *for (some) all* $x \in X$ *then the function* $\mu : X \to [0,\infty], x \mapsto \mu(x)$ *is constant.*

Proof. **Exercise 42**: For $x \in X$ and $n \geq 1$ let $\mu_n(x) = \frac{1}{n}\log m(B_n(x))$. Let n_k be such that $\limsup_n \mu(x) = \lim_k \mu_{n_k}(x)$. Then for all $r \geq 0$ we have $\mu(x) = \lim_k \mu_{n_k+r}(x)$ since $\mu_n \leq \frac{n+r}{n}\mu_{n+r}$. Suppose there is $y \in X$ such that $\mu(y) < \mu(x)$. Let $r = \delta(x,y)$. Then, we get for all $k \geq 1$

$$m(B_{n_k}(x)) \leq m(B_{n_k+r}(y)) \leq m(B_{n_k+2r}(x)).$$

Then, $\mu(x) = \lim_k \mu_{n_k}(y) \leq \mu(y)$ which is a contradiction. □

Theorem 22. *(Brooks theorem) Let* $\bigcup B_r(x) = X$ *and let* δ *be an intrinsic metric such that all distance balls are finite,* $m(X) = \infty$ *and* $c \equiv 0$*. Then,*

$$\lambda_0^{\mathrm{ess}}(L) \leq \frac{\mu^2}{4}.$$

In general, it is hard to determine the assumption whether the distance balls are finite with respect to a certain metric. The following corollary replaces the assumption by an assumption on the measure space.

Corollary 9. *Let* δ *be an intrinsic metric and assume that* m *gives infinite connected sets infinite measure. Then,* $\lambda_0^{\mathrm{ess}}(L) \leq \mu^2/4$*.*

Proof. In the case that there is an distance ball B_r with infinitely many vertices. By assumption $m(B_r) = \infty$ and thus $\mu = \infty$. Hence, the inequality is trivial. The other case follows from the theorem above. □

In the case where $\mathrm{Deg} \equiv C$ we have that ρ equals the regular graph metric. In particular, $b(x,y) > 0$ if and only if $\rho(x,y) = C^{-\frac{1}{2}}$. In this case we have even an better estimate.

Theorem 23. *(Brooks theorem - bounded version) Assume the graph* (b,c) *is connected,* $\mathrm{Deg} \leq C$*,* $m(X) = \infty$ *and* $c \equiv 0$*. Then, for* μ *with respect to the natural graph metric* d *we have*

$$\lambda_0^{\mathrm{ess}}(L) \leq C\Big(1 - \frac{1}{\cosh(\mu/2)}\Big).$$

8.2 The minimizing function

We introduce the sequence of test function which we use to apply Proposition 3. Fix $x_0 \in X$. Let $\alpha > 0$ and $n \geq 1$ define

$$g_{r,\alpha} : X \to \mathbb{R}, \quad x \mapsto \exp(\alpha(r \wedge (2r - \delta(x_0,x))))$$

Picture.

Clearly, $g_{\alpha,r}$ is constant on B_n and decreases exponentially outside of B_r. We show the following facts about $g_{r,\alpha}$.

Lemma 27. *For $\alpha > \mu/2$ and $r > 0$*

(a) *$g_{r,\alpha} \in \ell^2(X, m)$,*

(b) *$g_{k,\alpha}/\|g_{k,\alpha}\| \to 0$ weakly as $k \to \infty$ if $m(X) = \infty$ and $X = \bigcup_r B_r$,*

(c) *$g_{r,\alpha}(x) - g_{r,\alpha}(y) \leq \alpha(g^2_{r,\alpha}(x) + g^2_{r,\alpha}(y))^{\frac{1}{2}}\delta(x, y)$ for $x, y \in X$*

Proof. If $\mu = \infty$, then there is nothing to prove. (Note that this includes the case $m(B_r) = \infty$ for some $r \geq 0$.)
(a) Let $\alpha > \mu/2$ and $r \geq 0$. Then, since $g_{r,\alpha}$ are spherically symmetric we get for $r \in \mathbb{N}$

$$\|g_{r,\alpha}\|^2 \leq e^{2\alpha k} m(B_k + e^{4r\alpha} \sum_{k=r}^{\infty} e^{-2\alpha k}(m(B_{k+1}) - m(B_k))$$

$$\leq e^{2\alpha r} m(B_r) + e^{4ar}(1 - e^{-2\alpha}) \sum_{k=r}^{\infty} e^{-2\alpha k} m(B_k) < \infty$$

since $\alpha > \mu/2$ (i.e., there is $\beta \in (\mu, 2\alpha)$ such that $m(B_k) \leq e^{\beta k}$ for large k). For arbitrary $r \geq 0$ the statement follows from $g_{r,\alpha} \leq g_{k,\alpha}$ for $r \leq k$.
(b) Let $\alpha > \mu/2$, $\varphi \in \ell^2(X, m)$ with $\|\varphi\| = 1$ and $\varepsilon > 0$. Moreover, let $r > 0$ be such that

$$\|\varphi 1_{X \setminus B_r}\| \leq \varepsilon/2$$

and $R \geq r$ such that $m(B_r) \leq \frac{\varepsilon}{2} m(B_R)$, where this choice is possible since $m(X) = \infty$. Let $f_r = g_{r,\alpha}/\|g_{r,\alpha}\|$. Note that since $f_R \leq e^{\alpha R}/(e^{\alpha r} m(B_R)^{\frac{1}{2}}) = m(B_R)^{-\frac{1}{2}}$ we have

$$\|f_R 1_{B_r}\| \leq \frac{m(B_r)}{m(B_R)} \leq \frac{\varepsilon}{2}$$

and by definition $\|f_R\| = 1$. We estimate by the Cauchy-Schwarz inequality and

$$\langle \varphi, f_R \rangle = \langle \varphi 1_{B_r}, f_R \rangle + \langle \varphi 1_{X \setminus B_r}, f_R \rangle \leq \|\varphi\| \|f_R 1_{B_r}\| + \|\varphi 1_{X \setminus B_r}\| \|f_R\| \leq \varepsilon.$$

Hence, (f_k) converges weakly to zero.
(c) We estimate

$$|g_{r,\alpha}(x) - g_{r,\alpha}(y)| \leq \frac{|e^{2\alpha r - \alpha\delta(x,x_0)} - e^{2\alpha r - \alpha\delta(y,x_0)}|}{|\delta(x, x_0) - \delta(y, x_0)|} |\delta(x, x_0) - \delta(y, x_0)|.$$

Since $|(e^{a(s)} - e^{a(s+t)})/t| \leq a(e^{as} + e^{a(s+t)})/2$ for $a \geq 0$

$$\ldots \leq \frac{1}{2}(\alpha g_{r,\alpha}(x) + \alpha g_{r,\alpha}(y)) \leq \alpha(g_{r,\alpha}(x)^2 + g_{r,\alpha}(y)^2)^{\frac{1}{2}} \delta(x, y),$$

where we used $|\delta(x, x_0) - \delta(y, x_0)| \leq \delta(x, y)$ and $(s + t)^2 \leq 2s^2 + 2t^2$. □

Lemma 28. *If δ is the natural graph metric d, then for all $\alpha, r > 0$ and $x, y \in X$ with $b(x, y) \geq 0$*

$$g_{r,\alpha}(x) - g_{r,\alpha}(y) \leq \frac{(e^\alpha - 1)}{\sqrt{1 + e^{2\alpha}}}(g^2_{r,\alpha}(x) + g^2_{r,\alpha}(y))^{\frac{1}{2}} \delta(x, y).$$

Proof. **Exercise 43.** □

8.3 The key estimate

Lemma 29. *For $\alpha > \mu/2$ and $r > 0$ we have*

$$\frac{1}{2}\sum_{x,y\in X} b(x,y)(g_{r,\alpha}(x) - g_{r,\alpha}(y))^2 \leq \alpha^2 \|g_{r,\alpha}\|^2.$$

Proof. By Lemma 27 (c) we get using that δ is an intrinsic metric

$$\begin{aligned}
&\frac{1}{2}\sum_{x,y\in X} b(x,y)(g_{r,\alpha}(x) - g_{r,\alpha}(y))^2 \\
&\qquad \leq \frac{\alpha^2}{2}\sum_{x,y\in X} b(x,y)(g_{r,\alpha}^2(x) + g_{r,\alpha}^2(y))\delta(x,y)^2 \\
&\qquad = \frac{\alpha^2}{2}\Big(\sum_{x\in X} g_{r,\alpha}^2(x)\sum_{y\in X} b(x,y)\delta(x,y)^2 + \sum_{y\in X} g_{r,\alpha}^2(y)\sum_{x\in X} b(x,y)\delta(x,y)^2\Big) \\
&\qquad \leq \alpha^2 \sum_{x\in X} g_{r,\alpha}^2(x) m(x).
\end{aligned}$$

□

8.4 Proof of Brooks's Theorem

Lemma 30. *Let $f \in \mathcal{D}_{loc}^*$. For all $A \subseteq X$ and $\varphi \in \mathrm{Lip}$ supported on A, bounded by one and with Lipshitz constant one*

$$\frac{1}{2}\sum_X |d_b f\varphi|^2 \leq 2\sum_A |d_b f|^2 + 2\|f1_A\|^2.$$

Proof. For $h \in \mathcal{D}_{loc}^*$ with $\mathrm{supp} h \subseteq A$ we have

$$\begin{aligned}
&\frac{1}{2}\sum_X |d_b h|^2 \\
&\quad = \frac{1}{2}\Big(\sum_{(x,y)\in X\setminus A\times A} b(x,y)(h(x)-h(y))^2 + \sum_{(x,y)\in A\times X\setminus A} b(x,y)(h(x)-h(y))^2 \\
&\qquad\quad + \sum_{(x,y)\in A\times A} b(x,y)(h(x)-h(y))^2\Big) \\
&\quad = \sum_{(x,y)\in A\times X\setminus A} b(x,y)(h(x)-h(y))^2 + \frac{1}{2}\sum_{(x,y)\in A\times A} b(x,y)(h(x)-h(y))^2 \\
&\quad \leq \sum_A |d_b h|^2.
\end{aligned}$$

Since $\operatorname{supp}\varphi \subseteq A$, $\|\varphi\|_\infty \leq 1$ and $\varphi \in \mathrm{Lip}$, we have for all $f \in \mathcal{D}^*_{loc}$

$$\begin{aligned}(f\varphi(x) - f\varphi(y))^2 &\leq 2\varphi^2(y)(f(x)-f(y))^2 + 2f^2(x)(\varphi(x)-\varphi(y))^2\\ &\leq 2(f(x)-f(y))^2 + 2f(x)^2\delta(x,y)^2.\end{aligned}$$

Hence, by the above with $h = \varphi f$ (which is in $\mathcal{D}^*_{loc}$ by Lemma 21 (b)) and since δ is intrinsic

$$\frac{1}{2}\sum_X |d_b f\varphi|^2 \leq \sum_A |d_b f\varphi|^2 \leq 2\sum_A |d_b f|^2 + 2\|f1_A\|^2.$$

□

Lemma 31. *Assume $X = \bigcup_r B_r$ all distance balls are finite. Then, $g_{r,\alpha} \in D(Q)$ for all $\alpha > \mu/2$ and $r > 0$*

Proof. In order to show $g_{r,\alpha} \in D(Q)$ we have to show that $g_{r,\alpha}$ can be approximated with respect to the form norm $\|\cdot\|_Q$ by finitely supported functions. For $R > 0$ let $\eta_R = \eta_{1,B_R} = (1 - \inf_{y\in B_R}\delta(y,\cdot)) \vee 0$ as introduced earlier, where $B_R = B_R(x_0)$ is a ball about a fixed vertex x_0. Clearly, $\operatorname{supp}\eta_R = B_{R+1}$ and $\eta_R \in C_c(X)$ since we assumed that distance balls are finite. Moreover, by Lemma 22 we have $\eta_R \in \mathrm{Lip}$ with Lipshitz constant one and therefore $(1-\eta_R) \in \mathrm{Lip}$ which is bounded by one and has Lipshitz constant one. We show that we can approximate $g_{r,\alpha}$ by $\varphi_n = g_{r,\alpha}\eta_n \in C_c(X)$. By Lemma 30 we get with $A = X \setminus B_n$ since $\operatorname{supp}(1-\eta_n) = A$

$$\frac{1}{2}\sum_X |d_b(g_{r,\alpha} - \varphi_n)|^2 = \sum_X |d_b g_{r,\alpha}(1-\eta_n)|^2 \leq 2\sum_{X\setminus B_n} |d_b g_{r,\alpha}|^2 + 2\|g_{r,\alpha}1_{X\setminus B_n}\|^2.$$

By Lemma 27 (a) the functions $g_{r,\alpha}$ are in $\ell^2(X,m)$ and by Lemma 29, we have $\sum_X |d_b g_{r,\alpha}|^2 < \infty$. Hence, the right hand side converges to zero as $X = \bigcup_n B_n$. Clearly, $\|g_{r,\alpha} - \varphi_n\| \leq \|g_{r,\alpha}1_{X\setminus B_n}\| \to 0$ and, thus, φ_n converges with respect to $\|\cdot\|_Q = \sqrt{Q(\cdot) + \|\cdot\|^2}$. □

Proof of Brooks's theorem. By the lemma above $g_{r,\alpha} \in D(Q)$ for $\alpha > \mu/2$ and $r > 0$. By Lemma 29 we have for $f_k = g_{r,\alpha}/\|g_{k,\alpha}\|$

$$Q(f_k) \leq \alpha^2.$$

We assumed $m(X) = \infty$, so by Lemma 22 (b) the functions f_k converge weakly to zero. By Proposition 3 we have for all $\alpha > \mu/2$

$$\lambda_0^{\mathrm{ess}}(L) \leq \liminf_{k\to\infty} Q(f_k) \leq \alpha^2.$$

Hence, the statement follows. □

Proof of Brooks's theorem - the bounded case. If $\mathrm{Deg} \leq C$, then Q is bounded by Theorem 8. Let $b_C = b/C$. Then, the natural graph metric d is an intrinsic metric for $Q_C = \frac{1}{C}Q$, i.e.,

$$\sum_{y\in X} b_C(x,y)d(x,y)^2 = \frac{1}{C}\sum_{y\in X} b(x,y) = \frac{1}{C}m(x)\mathrm{Deg}(x) \leq m(x).$$

Let $g_{r,\alpha}$ defined with respect to d. Then, $g_{r,\alpha} \in \ell^2(X,m)$ for $\alpha > \mu/2$ by Lemma 27 (a). Moreover, since Q, Q_C are bounded we have $g_{r,\alpha} \in \ell^2(X,m) = D(Q) = D(Q_C)$. Hence, by Lemma 28 with $\beta = \frac{(e^\alpha-1)^2}{1+e^{2\alpha}} = 1 - 2e^\alpha/(1+e^\alpha) = 1 - 1/\cosh(\alpha/2)$

$$\frac{1}{C}Q(g_{r,\alpha}) = Q_C(g_{r,\alpha}) \leq \frac{\beta}{2}\sum_{x,y\in X} b_C(x,y)(g_{r,\alpha}^2(x) + g_{r,\alpha}^2(y))d(x,y)^2 \leq \beta\|g_{r,\alpha}\|^2.$$

since d is an intrinsic metric for Q_C. Thus, by Proposition 3

$$\lambda_0^{\mathrm{ess}}(L) \leq C\Big(1 - \frac{1}{\cosh(\mu/2)}\Big).$$

□

8.5 Applications

Let $b : X \times X \to \{0,1\}$, $c \equiv 0$ and $m \equiv 1$. Thus, we are concerned with the operator

$$\Delta\varphi(x) = \sum_{y\sim x}(\varphi(x) - \varphi(y)).$$

Theorem 24. *Let G be a graph such that* $\mathrm{Deg} \leq k+1$. *Then, $\mu \leq \log k$ with respect to the natural graph metric and*

$$\lambda_0^{\mathrm{ess}}(\Delta) \leq k + 1 - 2\sqrt{k}.$$

Proof. Let T be a spanning tree of G which leaves the distance relation to a fixed vertex invariant. (A spanning tree is a connected subgraph with the same vertex set that is a tree.) This can easily be achieved as follows: One removes all edges which connect vertices within a sphere. Further one removes inductively all edges which connect a vertex in a sphere to vertices in the previous sphere except for one edge. Moreover, we can embed T in a k-regular tree T_k. We obtain,

$$\mu(G) = \mu(T) \leq \mu(T_k) = \limsup_{n\to\infty}\frac{1}{n}\log m(B_n) = \limsup_{n\to\infty}\frac{1}{n}\log\sum_{j=1}^{n}k^j = \log k.$$

Since cosh is monotone increasing the statement follows from the bounded version of Brooks theorem by direct calculation. □

Chapter 9

Isoperimetric constants and lower bounds

For non-compact Riemannian manifolds M without boundary the Cheeger constant is defined as

$$h = \inf \frac{\text{Area}(\partial U)}{\text{Vol}(U)},$$

where the infimum is taken over all compact submanifolds U with smooth boundary ∂U. Then,

$$\lambda(\Delta_M) \geq \frac{h^2}{4}.$$

We prove a similar estimate for graphs. Let (b, c) be a graph over (X, m). Let Q be the corresponding form and L the associated operator. We first treat the case $c \equiv 0$ and discuss later how to incorporate the case $c \not\equiv 0$.

For a finite subset $W \subseteq X$ we define

$$\partial W = (W \times X \setminus W) \cup (X \setminus W \times W) \subset X \times X$$

and

$$|\partial W| = \frac{1}{2} \sum_{(x,y) \in \partial W} b(x, y).$$

Of particular importance will be the measure n given by

$$n(x) = \sum_{y \in X} b(x, y) = |\partial\{x\}|.$$

Define the Cheeger constant as of a subset $U \subseteq X$

$$\alpha_U = \inf_{\emptyset \neq W \subseteq U \text{ finite}} \frac{|\partial W|}{n(W)}$$

We have $\alpha_U \leq 1$ by $\partial W \subseteq \bigcup_{x\in W} \partial\{x\}$. For $U = X$ denote $\alpha = \alpha_X$.

Example Let $b : X \times X \to \{0, 1\}$. Then $|\partial W| = \#\partial W$ that is the number of edges leaving W. Since $n = \deg$, we have that $n(W) = \sum_W \deg$ is twice the number of edges in W plus once the number of edges leaving W.

Let

$$\underline{D}_U = \inf_{x\in U} \mathrm{Deg}(x).$$

Note that since we assumed $c \equiv 0$ we have $\mathrm{Deg} = \frac{1}{m(x)} \sum_{y\in X} b(x, y)$. For $U = X$ denote $\underline{D} = \underline{D}_X$.

Our goal is the following theorem.

Theorem 25. *(Cheeger inequality)*

$$\lambda_0(L) \geq (1 - \sqrt{1 - \alpha^2})\underline{D}.$$

Note that $(1 - \sqrt{1 - \alpha^2}) \geq \frac{\alpha^2}{2}$ by the Taylor series expansion we get $\sqrt{1 - s} = 1 - s/2 - s^2/8 \ldots$.

9.1 Co-area formula

Let $c \equiv 0$.

Theorem 26. *(Co-Area formulae) Let $f : X \to \mathbb{R}$ be given and for $t \in \mathbb{R}$ define*

$$\Omega_t = \{x \in X \mid f(x) > t\}.$$

Then,

$$\frac{1}{2} \sum_{x,y\in X} b(x, y)|f(x) - f(y)| = \int_{-\infty}^{\infty} |\partial\Omega_t| dt$$

and

$$\sum_{x\in X} |f(x)|m(x) = \int_{-\infty}^{\infty} m(\Omega_t)dt.$$

Proof. For $x, y \in X$ with $x \not\in y$ we define the interval $I_{x,y}$ by

$$I_{x,y} := [f(x) \wedge f(y), f(x) \vee f(y))$$

and let $|I_{x,y}| = |f(x)-f(y)|$ be the length of $I_{x,y}$. Let $1_{x,y} := 1_{I_{x,y}}$. Then, $(x, y) \in \partial\Omega_t$ if and only $t \in I_{x,y}$. Thus,

$$|\partial\Omega_t| = \frac{1}{2} \sum_{x,y\in X} b(x, y)1_{x,y}(t).$$

We calculate

$$\begin{aligned}\int_{-\infty}^{\infty} |\partial\Omega_t| dt &= \frac{1}{2}\int_{-\infty}^{\infty} \sum_{x,y\in X} b(x,y) 1_{x,y}(t) dt \\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y) \int_{-\infty}^{\infty} 1_{x,y}(t) dt \\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y)|f(x)-f(y)|.\end{aligned}$$

Similarly we have $x \in \Omega_t$ if and only if $1_{(t,\infty)}(f(x)) = 1$. Thus, we can calculate

$$\begin{aligned}\int_{-\infty}^{\infty} m(\Omega_t) dt &= \int_{-\infty}^{\infty} \sum_{x\in X} m(x) dt \\ &= \int_{-\infty}^{\infty} \sum_{x\in X} m(x) 1_{(t,\infty)}(f(x)) dt \\ &= \sum_{x\in X} m(x) \int_{-\infty}^{\infty} 1_{(t,\infty)}(f(x)) dt \\ &= \sum_{x\in X} |f(x)| m(x).\end{aligned}$$

This finishes the proof. □

Remark. If $f : X \to [0,\infty)$, then it suffices to take the integral from zero to infinity.

9.2 An isoperimetric inequality

Let $\nu : X \to (0,\infty)$ be the measure as above and for $\varphi \in C_c(X)$ let

$$\|\varphi\|_\nu = \Big(\sum_X \varphi^2\nu\Big)^{\frac{1}{2}}.$$

Proposition 5. *Let $U \subseteq X$. Then for all $\varphi \in C_c(U)$*

$$Q(\varphi)^2 - 2\|\varphi\|_n^2 Q(\varphi) + \alpha_U^2 \|\varphi\|_n^4 \le 0$$

Proof. Denote A by

$$A = \frac{1}{2}\sum_{x,y\in X} b(x,y)|\varphi^2(x) - \varphi^2(y)| = \frac{1}{2}\sum_{x,y\in X} b(x,y)|\varphi(x)-\varphi(y)||\varphi(x)+\varphi(y)|.$$

By Cauchy-Schwarz inequality

$$\begin{aligned}A^2 &= Q(\varphi)\Big(\frac{1}{2}\sum_{x,y\in X} |\varphi(x)+\varphi(y)|^2\Big) \\ &= Q(\varphi)\Big(\frac{1}{2}\sum_{x,y\in X} b(x,y) 2\varphi^2(x) + 2\varphi^2(y) - |\varphi(x)-\varphi(y)|^2\Big) \\ &= Q(\varphi)(2\|\varphi\|^4 - Q(\varphi)).\end{aligned}$$

On the other hand, we can use the first and the second co-area formula (with $f = \varphi^2$) to estimate

$$A = \int_0^\infty |\partial\Omega_t| dt \geq \alpha \int_0^\infty m(\Omega_t) dt = \alpha \sum_{x\in X} \varphi(x)^2 n(x) = \alpha \|\varphi\|_n^2.$$

Combining the two estimates on A, we obtain

$$Q(\varphi)\big(2\|\varphi\|_n^2 - Q(\varphi)\big) \geq \alpha^2 \|\varphi\|_n^4.$$

This yields the desired result. $\square$

9.3 Lower bounds

Let $(b, 0)$ be a graph over (X, m). For $U \subseteq X$ let Q_U be the closure of Q on $C_c(U) \subseteq C_c(X)$, i.e., $D(Q_U) = \overline{C_c(U)}^{\|\cdot\|_Q}$. Then, $Q(\varphi) = Q_U(\varphi)$ for all $\varphi \in D(Q_U)$. Let L_U be the operator associated to Q_U.

Moreover, we write

$$a \leq Q_U \leq b$$

for $a, b \in \mathbb{R}$, whenever

$$a\|f\|^2 \leq Q_U(f) \leq b\|f\|^2,$$

where $\|\cdot\| = \|\cdot\|_m$. Recall that $\underline{D}_U = \inf_{x\in U} \mathrm{Deg}(x)$ and define

$$\overline{D}_U = \sup_{x\in U} \mathrm{Deg}(x).$$

Theorem 25 follows from the following theorem.

Theorem 27. *For $U \subseteq X$*

$$\underline{D}_U(1 - \sqrt{1-\alpha_U^2}) \leq Q_U \leq \overline{D}_U(1 + \sqrt{1-\alpha_U^2})$$

In particular, $\sigma(L_U) \subseteq [\underline{D}_U(1 - \sqrt{1-\alpha_U^2}), \overline{D}_U(1 + \sqrt{1-\alpha_U^2})]$. Moreover,

$$\underline{D}_U(1 - \sqrt{1-\alpha_U^2}) \leq \lambda_0(L_U) \leq \overline{D}_U \alpha_U.$$

Proof. Let $\varphi \in C_c(U)$ with $\|\varphi\|_n = 1$. Then, the isoperimetric inequality gives

$$Q(\varphi)^2 - 2Q(\varphi) + \alpha_U^2 \leq 0$$

and, therefore,

$$1 - \sqrt{1-\alpha_U^2} \leq Q(\varphi) \leq 1 + \sqrt{1-\alpha_U^2}.$$

As for all $\varphi \in C_c(U)$ we get using $\mathrm{Deg} = n/m$

$$\|\varphi\|_n = \sum_U \varphi^2 n = \sum_U \varphi^2 m \mathrm{Deg}$$

that

$$\underline{D}_U \|\varphi\|_m^2 \leq \|\varphi\|_n^2 \leq \overline{D}_U \|\varphi\|_m^2.$$

The inclusion of spectrum as a set follows from Corollary 3. Finally, note that for finite $W \subseteq U$

$$Q(1_W) = \sum_{x \in W} \sum_{y \notin W} b(x,y) = |\partial W|.$$

Hence,

$$\frac{Q_U(1_W)}{\|1_W\|} = \frac{|\partial W|}{m(W)} = \frac{|\partial W|}{n(W)} \frac{\sum_W \mathrm{Deg} m}{m(W)} \leq \overline{D}_U \frac{|\partial W|}{n(W)}.$$

Therefore,

$$\begin{aligned}\lambda_0(L_U) &= \inf_{\varphi \in C_c(U), \varphi \neq 0} \frac{Q_U(\varphi)}{\|\varphi\|} \leq \inf_{\emptyset \neq W \subseteq U \text{ finite}} \frac{Q_U(1_W)}{\|1_W\|} \\ &\leq \overline{D}_U \inf_{\emptyset \neq W \subseteq U \text{ finite}} \frac{|\partial W|}{n(W)} = \overline{D}_U \alpha_U.\end{aligned}$$

Thus the final statement follows from the spectral inclusion. □

Examples(a) For the operator Δ on $\ell^2(X)$ we get

$$(1 - \sqrt{1 - \alpha^2}) \inf_{x \in X} \deg(x) \leq \lambda_0(\Delta) \leq \alpha \sup_{x \in X} \deg(x)$$

.

(b) For $\widetilde{\Delta}$ on $\ell^2(X, \deg)$ we get since $\mathrm{Deg} \equiv 1$ that

$$\sigma(\widetilde{\Delta}) \subseteq [1 - \sqrt{1 - \alpha^2}, 1 + \sqrt{1 - \alpha^2}]$$

and

$$(1 - \sqrt{1 - \alpha^2}) \leq \lambda_0(\widetilde{\Delta}) \leq \alpha$$

Thus we have $\lambda_0(\widetilde{\Delta})$ if and only if $\alpha = 0$.

Corollary 10. *If L is bounded, then $\lambda_0 = 0$ if and only if $\alpha_X = 0$.*

Proof. If L is bounded, then Deg is bounded and $\overline{D} < \infty$. If $\lambda_0 = 0$, then $0 \leq \underline{D}_X(1 - \sqrt{1 - \alpha_X^2}) \leq \lambda_0 = 0$ which implies $\alpha = 0$. On the other hand, if $\alpha = 0$ then, $0 \leq \lambda_0 \leq \overline{D}_X \alpha_X = 0$. □

9.4 Non vanishing potentials

We will use a trick that allows us to prove the statement for non-vanishing potentials. Let (b, c) be a graph over (X, m).

Let $\dot{X} = X \times \{0, 1\}$. We can consider X as a subset of $\dot{X}$ by the embedding

$$X \hookrightarrow \dot{X}, \; x \mapsto (x, 0).$$

So, we think of the elements $x_0 = (x, 0)$ being a vertex in X and of $x_1 = (x, 1)$ as a virtual vertex related to x_0 at infinity. In this sense $C_c(X) \subseteq C_c(\dot{X})$.

Define a symmetric $\dot{b}$ with zero diagonal on $\dot{X} \times \dot{X}$ be such that for $x, y \in X$, $x \neq y$,

$$\begin{aligned} \dot{b}(x_0, y_0) &= \dot{b}(y_0, x_0) = b(x, y), \\ \dot{b}(x_0, x_1) &= \dot{b}(x_1, x_0) = c(x), \end{aligned}$$

and zero otherwise. Moreover, let $\dot{c} \equiv 0$ and let $\dot{m}$ be m on X and arbitrary (e.g. zero) otherwise. By the embedding $X \hookrightarrow \dot{X}$ we have $\ell^2(X, m) \subseteq \ell^2(\dot{X}, \dot{m})$.

Then, the corresponding form $\dot{Q}$ satisfies

$$\dot{Q}(\varphi) = Q(\varphi)$$

for $\varphi \in C_c(X) \subseteq C_c(\dot{X})$. Furthermore, restricting $\dot{Q}$ to $C_c(X)$ and taking the closure we find that

$$D(\dot{Q}_X) = D(Q) \quad \text{and} \quad \dot{Q}_X(\varphi) = Q(\varphi).$$

The vertex degrees $\dot{\mathrm{Deg}}$ and Deg agree on X, i.e.,

$$\dot{\mathrm{Deg}}(x) = \frac{1}{\dot{m}(x)} \Big(\sum_{y \in X} \dot{b}(x, y) + \sum_{y \in \dot{X} \setminus X} \dot{b}(x, y) \Big) = \frac{1}{m(x)} \Big(\sum_{y \in X} b(x, y) + c(x) \Big) = \mathrm{Deg}(x).$$

Hence, $\underline{\dot{D}}_X = \underline{D}_X$. The Cheeger constant $\dot{\alpha}_X$ of $X \subseteq \dot{X}$ satisfies

$$\dot{\alpha}_X = \inf_{\emptyset \neq W \subseteq X \subseteq \dot{X}, \, \text{finite}} \frac{|\dot{\partial} W|}{\dot{n}(W)},$$

where

$$|\dot{\partial} W| = \sum_{\dot{x} \in W, \dot{y} \in \dot{X} \setminus W} \dot{b}(\dot{x}, \dot{y}) = \sum_{x \in W} \Big(\sum_{y \in X \setminus W} b(x, y) + c(x) \Big) = |\partial W| + c(W)$$

and $\dot{n} = n$ on X.

Hence, Theorem 27 holds for $\dot{Q}_X$ with $\dot{\alpha}_X$ and $\underline{\dot{D}}_X = \underline{D}_X$. We get the following theorem for a graph (b, c) over (X, m) with non-vanishing potential.

Theorem 28. *For $U \subseteq X$ $\sigma(L_U) \subseteq [\underline{D}_U(1-\sqrt{1-\dot\alpha_U^2}), \overline{D}_U(1+\sqrt{1-\dot\alpha_U^2})]$ and*

$$\underline{D}_U(1-\sqrt{1-\dot\alpha_U^2}) \le \lambda_0(L_U) \le \overline{D}_U\dot\alpha_U.$$

In order to determine whether $\dot\alpha_U > 0$ we can consider the following constant:

Lemma 32. *$\dot\alpha_U > 0$ if and only if*

$$\beta_U := \inf_{\emptyset \neq W \subseteq U, \textit{finite}} \frac{|\partial W| + c(W)}{n(W)} > 0.$$

Proof. **Exercise 44**. □

From now on we denote the Cheeger constant $\dot\alpha_U$ introduced in the previous section with slight abuse of notation by α_U, $U \subseteq X$.

9.5 Lower bounds on the essential spectrum

Let (b,c) be a graph over (X,m).

Let $\mathcal{K}$ be the set of finite subsets of X. For a function $F : \mathcal{K} \to \mathbb{R}$ we say that F converges to $r \in \mathbb{R}$ if for all $\varepsilon > 0$ there is a set $K \in \mathcal{K}$ such that for all $L \in \mathcal{K}$ with $K \subseteq L$ we have $|F(L) - r| \le \varepsilon$. In this case we write $\lim_{K\in\mathcal{K}} F(K) = r$.

Note that $0 \le \alpha_{X\setminus K} \le \alpha_{X\setminus L} \le 1$ for all $K \subseteq L$. Thus, the limit

$$\alpha_\infty = \lim_{K\in\mathcal{K}} \alpha_{X\setminus K}$$

exist and is in $[0,1]$. Clearly, $\alpha_\infty \ge \alpha$. Similarly, $0 \le \underline{D}_{X\setminus K} \le \underline{D}_{X\setminus L}$ for all $K \subseteq L$ and we find that

$$\underline{D}_\infty = \lim_{K\in\mathcal{K}} \underline{D}_{X\setminus K}$$

exists in $[0,\infty]$.

Theorem 29. *Let (b,c) be locally finite. Then,*

$$\sigma_{\mathrm{ess}}(L) \subseteq [\underline{D}_\infty(1-\sqrt{1-\alpha_\infty^2}), \overline{D}_\infty(1+\sqrt{1-\alpha_\infty^2})],$$

where the lower bound is zero if $\alpha_\infty = 0$ and $\underline{D} = \infty$. Moreover,

$$\underline{D}_\infty(1-\sqrt{1-\alpha_\infty^2}) \le \lambda_0^{\mathrm{ess}}(L) \le \overline{D}_\infty\alpha_\infty,$$

where the upper bound is ∞ if $\alpha_\infty = 0$ and $\overline{D} = \infty$.

Proof. Let $K \in \mathcal{K}$ and be the operator arising from $Q_{X\setminus K}$. Since the graph is locally finite the operators L and $L_{X\setminus K}$ differ only in finitely many matrix elements and the operator $L - L_{X\setminus K}$ is a finite dimensional operator. By Theorem 14 the operators L and $L_{X\setminus K}$ have the same essential spectrum. Thus, then inclusion statement for the spectrum follows from Theorem 27. We now turn to the upper bound on λ_0^{ess}. Let $K_n \in \mathcal{K}$ such that $K_n \subseteq K_{n+1}$, $n \in \mathbb{N}$, and $X = \bigcup_{n\geq 1} K_n$ choose normalized functions $f_n \in C_c(X \setminus K_n)$ such that $|(Q - \lambda_0(L_{X\setminus K_n}))(f_n, f_n)| \leq 1/n$ (this is possible by Corollary 3). As f_n is supported on $X \setminus K_n$ the sequence (f_n) is a weak null-sequence. Hence, by Proposition 3 we get

$$\lambda_0^{\mathrm{ess}}(L) \leq \liminf_{n\to\infty} Q(f_n) = \liminf_{n\to\infty} \lambda_0(L_{X\setminus K_n}) \leq \liminf_{n\to\infty} \overline{D}_{X\setminus K}\alpha_{X\setminus K} = \overline{D}_\infty \alpha_\infty.$$

□

Exercise 45: Prove the statement by replacing the local finiteness assumption by $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$.

Corollary 11. *Assume the graph is locally finite. If* $D_\infty := \underline{D}_\infty = \overline{D}_\infty$. *Then,* $\sigma_{\mathrm{ess}}(L) = \{D_\infty\}$ *if and only if* $\alpha_\infty = 1$.

Proof. Assume $\sigma_{\mathrm{ess}}(L) = \{D_\infty\}$, then $\lambda_0^{\mathrm{ess}} = D_\infty$. By $D_\infty(1 - \sqrt{1-\alpha_\infty}) \leq \lambda_0^{\mathrm{ess}} \leq D_\infty \alpha_\infty$ we get $1 - \alpha_\infty \leq (1-\alpha_\infty)^2$. Since $\alpha_\infty \leq 1$ we obtain $\alpha_\infty = 1$. On the other hand if $\alpha_\infty = 1$, then the spectral inclusion implies $\lambda_0^{\mathrm{ess}} = \{D_\infty\}$. □

Example. For $\widetilde{\Delta}$ on $\ell^2(X, \deg)$ we have that $\sigma_{\mathrm{ess}}(\widetilde{\Delta}) = \{1\}$ iff $\alpha_\infty = 1$.

Corollary 12. *Assume the graph is locally finite and* $\alpha_\infty > 0$. *Then,* $\sigma_{\mathrm{ess}}(L) = \infty$ *if and only if* $\underline{D}_\infty = \infty$.

Proof. The statement follows from $\underline{D}_\infty(1 - \sqrt{1-\alpha_\infty^2}) \leq \lambda_0^{\mathrm{ess}} \leq \overline{D}_\infty \alpha_\infty$. □

Example. For Δ on $\ell^2(X)$ with $\alpha_\infty > 0$ we have that $\sigma_{\mathrm{ess}}(\widetilde{\Delta}) = \emptyset$ iff $\deg(x_n) \to \infty$ for all (x_n) with $x_n \sim x_{n+1}$, $n \geq 1$.

9.6 Lower bounds for Cheegers constant

Let (b, c) be a graph over (X, m) and recall that n was defined as

$$n(x) = \sum_{y\in X} b(x, y).$$

Fix a vertex $x_0 \in X$. Denote by S_r, $r \geq 0$ the distance spheres about x_0 with respect to the natural graph metric. As in Section 5.5 let

$$b_\pm : X \to [0, \infty), \quad b_\pm(x) = \sum_{y\in S_{r\pm 1}} b(x, y), \quad x \in S_r$$

and define

$$K : X \to [0, \infty), \quad x \mapsto \frac{b_-(x) - b_+(x)}{n(x)}$$

The function K is referred to as *mean curvature* of the graph.
Theorem 30. $\alpha \geq -\sup_{x \in X} K(x)$

Proof. Let $r : X \to [0, \infty)$, $x \mapsto d(x, x_0)$. Let $\mathcal{L}$ be the formal operator with respect to $(b, 0)$ over (X, n). Then, for $x \in S_r$, $y \in S_{r\pm1}$, we have $r(x) - r(y) = \mp 1$ and, therefore,

$$\begin{aligned}\mathcal{L}r(x) &= \frac{1}{n(x)} \sum_{y \in S_{r-1}} b(x,y)(r(x) - r(y)) - \frac{1}{n(x)} \sum_{y \in S_{r+1}} b(x,y)(r(y) - r(x)) \\ &= \frac{b_-(x) - b_+(x)}{n(x)} = K(x).\end{aligned}$$

Thus, $r \in \mathcal{F}$. Let $C := -\sup_{x \in X} K(x)$ and $W \subseteq X$ finite. Then, using Green's formula

$$\begin{aligned}Cn(W) &\leq \sum_W (-\mathcal{L}r)n = -\sum_X 1_W(\mathcal{L}r)n \\ &= -\frac{1}{2} \sum_{x,y \in X} b(x,y)(r(x) - r(y))(1_W(x) - 1_W(y)) \\ &\leq \frac{1}{2} \sum_{x,y \in X} b(x,y)|r(x) - r(y)||1_W(x) - 1_W(y)| \\ &\leq \frac{1}{2} \sum_{x,y \in X} b(x,y)|1_W(x) - 1_W(y)|^2 \\ &= |\partial W|.\end{aligned}$$

Thus, $\alpha \geq c$. □

Example Let $b : X \times X \to \{0, 1\}$, $c \equiv 0$ and $m \equiv 1$ be a tree, i.e., for some x_0 we have $b_-(x_0) = 0$ and $b_-(x) = 1$ for $x \neq x_0$. Hence, $K = \frac{1-b_+}{1+b_+}$. If $b_+ \geq k$

$$\alpha \geq \inf_{x \in X} \frac{b_+(x) - 1}{b_+(x) + 1} = 1 - \frac{2}{k+1}$$

which shows that the estimate is sharp for k-regular trees

$$\lambda_0(\Delta) \geq (k+1)\Big(1 - \sqrt{1 - (\frac{k-1}{k+1})^2}\Big) = k + 1 - 2\sqrt{k}.$$

Thus, Moreover, if $\underline{D} = \infty$, then $\sigma_{\mathrm{ess}}(\Delta) = \emptyset$.

Chapter 10

Tessellations

Let X be countable, $m \equiv 1$ and $b : X \times X \to \{0, 1\}$. A graph is called *planar* if X can be embedded into a surface $\mathcal{S}$ homeomorphic to $\mathbb{R}^2$ or $\mathbb{S}^2$ such that all $x, y \in X$, $x \sim y$ can be joined by continuous curves without intersection. We identify the graph with its embedding and call the connecting curves edges. We denote this set by E. Moreover, we call the closures of the connected components of $\mathcal{S} \setminus \bigcup E$ the *faces* of the graph and denote them by F. We call a face a *polygon* if it is homeomorphic to the unit disc $\mathbb{D} = \{z \in \mathbb{R}^2 \mid |z| \leq 1\}$.

In the following we denote a planar graph by the triple $G = (X, E, F)$. We say that G is *locally finite* if for every point in $\mathcal{S}$ there exists an open neighborhood of this point that intersects with only finitely many edges.

A graph is called a *tessellation* if

(T1) Every edge is included in two faces.

(T2) Every two faces are either disjoint or intersect in one edge or one vertex.

(T3) Every face is a polygon.

For the rest of this chapter we are concerned with locally finite tessellations.

10.1 Curvature

Let $G = (X, E, F)$ be a planar graph that is a tessellation. An important geometric quantity is the curvature of a graph. For a face f we denote by $\deg(f)$ the number of vertices contained in f, i.e., we have

$$\deg(f) := \#\{x \in X \mid x \in f\} = \#\{e \in E \mid e \subset f\}$$

Let

$$\kappa : X \to \mathbb{R}, \quad x \mapsto 1 - \frac{\deg(\mathrm{x})}{2} + \sum_{f \in F, x \in f} \frac{1}{\deg(f)}.$$

This can be motivated as follows: In Euclidian geometry a regular polygon is a cyclic, equiangular polygon, i.e., its corners lie on a circle and its corner angles are all equal. Let a regular polygon with n corners be given. Then, the corner angle $\alpha(n)$ can be calculated as follows: Walking around the polygon once yields an angle of 2π. To do so one passes n corners each with an angle $\pi - \alpha(n)$, i.e.,

$$2\pi = n(\pi - \alpha(n))$$

Thus,

$$\alpha(n) = \frac{2\pi(n-2)}{2n}$$

On the other hand, we have

$$\begin{aligned} 2\pi\kappa(x) &= 2\pi\Big(1 - \frac{\mathrm{deg(x)}}{2} + \sum_{f\in F, x\in f} \frac{1}{\deg(f)}\Big) \\ &= 2\pi - \sum_{f\in F, x\in f} \frac{2\pi(\deg(f) - 2}{2\deg(f)} \\ &= 2\pi - \sum_{f\in F, x\in f} \alpha(\deg(f)). \end{aligned}$$

A subset $W \subseteq X$ induces a subgraph by letting the edges E_W be the ones whose starting and end vertices are in W. The faces are the ones obtained from the embedding and are denote by F_W. Note that the face set of F_W can strongly differ from F. We denote the curvature function of the graph $G_W = (W, E_W, F_W)$ by κ_W.

Lemma 33. *(Euler's formula) For a finite and connected graph W we have*

$$|W| - |E_W| + |F_W| = 2$$

Proof. By induction over $|W|$. □

Theorem 31. *(Gauß-Bonnet) Let $W \subseteq X$ be connected and finite. Then,*

$$\kappa_W(W) = \sum_{x\in W} \kappa_W(x) = |W| - |E_W| + |F_W| = 2$$

Proof. **Exercise 46**. □

Corollary 13. *If $\kappa \leq 0$, then X is infinite.*

Proof. Assume X is a finite tessellation. Then, $\kappa(X) = 2$ which is a contradiction to $\kappa \leq 0$. □

For a set $W \subseteq X$ let

$$\partial_F W = \{f \in F \mid f \cap W \neq \emptyset, f \cap X \setminus W = \emptyset\}$$

and

$$\deg_W(f) = \#(f \cap X \setminus W)$$

A set $W \subseteq X$ is called *simply connected* if W and $X \setminus W$ are connected.

Proposition 6. *Let W finite and simply connected. Then,*

$$\kappa(W) = 1 - \frac{|\partial W|}{2} + \sum_{f \in \partial_F W} \frac{\deg_W(f)}{\deg(f)}.$$

Proof. We start with two important formulas

$$\sum_{x \in X} \deg(x) = 2|E_W| + |\partial W|$$

Resorting the sum gives

$$\sum_{x \in W} \sum_{f \in F, x \in f} \frac{1}{\deg(f)} = |F_W| - 1 + \sum_{f \in \partial_F W} \frac{\deg_W(f)}{\deg(f)}$$

Hence,

$$\begin{aligned} \kappa(W) &= |W| - \sum_{x \in W} \frac{\deg(x)}{2} + \sum_{x \in W} \sum_{f \in F, x \in f} \frac{1}{\deg(f)} \\ &= |W| - |E_W| - \frac{|\partial W|}{2} + |F_W| - 1 + \sum_{f \in \partial_F W} \frac{\deg_W(f)}{\deg(f)}. \end{aligned}$$

and the statement follows from Euler's formula. □

Proposition 7. *(Absence of cut locus) If the tessellation is p, q regular then there every vertex in S_n is adjacent to a vertex in S_{n+1}, $n \geq 0$.*

Proof. **Exercise 47*** □

10.2 Volume growth of regular tessellations

For this section we restrict ourselves to regular tessellations, i.e., there are $p, q \geq 3$ such that $\deg(x) = p$ for $x \in X$ and $\deg(f) = q$ for $f \in F$.

We further restrict ourselves to the case of negative curvature $\kappa < 0$, i.e., $\frac{1}{p} - \frac{1}{2} + \frac{1}{q} < 0$. Moreover, as for odd q there are more case to distinguish we restrict ourselves to even q.

We compute the volume growth of a tessellation with respect to the natural graph metric.

Let $N = \frac{q-2}{2}$ and $P = P_{p,q} : \mathbb{C} \to \mathbb{C}$ be given by

$$P_{p,q}(z) = z^{N+1} - (p-2)\sum_{k=1}^{N} z^k + 1.$$

We will see that the largest root of the polynomial encodes the volume growth of the tessellation.

We will prove the following auxiliary statement.

Proposition 8. *Let $p, q \geq 3$ be such that $\frac{1}{p} - \frac{1}{2} + \frac{1}{q} < 0$ and q is even. Then, if $z \in \mathbb{C}$ is a root, so is $1/z$. Moreover, $N-1$ roots of $P_{p,q}$ lie on the complex unit circle and two roots are real and the larger one lies in $(1, p-1)$.*

Proof. Clearly $z = 0$ is no root of P. For $z \neq 0$ we have $P(z) = z^{N+1}P(1/z)$. Thus $P(z) = 0$ implies $P(1/z) = 0$. Let $\mathbb{S}^1$ be the complex unit circle. Denote

$$Q(z) = (z-1)P(z) = z^{N+2} - (p-1)z^{N+1} + (p-1)z - 1$$

and $z \in \mathbb{S}^1$ is a root of Q if and only if

$$z^{N+1} = \frac{-(p-1)z + 1}{z - (p-1)}.$$

Let $c_1, c_2 : [0, 2\pi] \to \mathbb{S}^1$ be the closed curves given by

$$c_1(t) = e^{i(N+1)t}, \quad c_2(t)\frac{-(p-1)e^{it} + 1}{e^{it} - (p-1)} = e^{it}\frac{(p-1) - e^{-it}}{(p-1) - e^{it}}.$$

which start and end in $z = 1$. (They are closed curves a $[0, 2\pi] \to \mathbb{C}$, $t \mapsto e^{it}$ is a closed curve.) The of winding numbers of c_1 and c_2 are $\mathrm{ind}_{c_1}(0) = N+1$ and $\mathrm{ind}_{c_2}(0) = 1$. (**Exercise 48**.) Therefore the two curves intersect in at least $N-1$ different values $t_1 < t_2 < \ldots < t_{N-1}$ of the open interval $(0, 2\pi)$ which corresponds to $N-1$ different zeros of Q and thus of P. Note also that

$$P(p-1) = \frac{1}{(p-1)}Q(p-1) = \frac{1}{(p-1)}\Big((p-1)^2 - 1\Big) > 0$$

and

$$P(1) = 2 - N(p-2) = pq\Big(\frac{1}{p} - \frac{1}{2} + \frac{1}{q}\Big) < 0.$$

Thus P must have a root in $(1, p-1)$. Since $P(z) = z^{N+1}P(1/z)$ there must be another root in $(0, 1)$. $\square$

Proposition 9. *If $q = 4, 6$*

$$\lambda_{\max}(p,q) = \frac{p}{2} - \frac{2}{q-2} + \sqrt{\Big(\frac{p}{2} - \frac{2}{q-2}\Big)^2 - 1}$$

Proof. In the case $q = 4$ we have $N = 1$ and

$$P_{p,4}(z) = z^2 - (p-2) + 1$$

and for $q = 6$ we have $N = 2$

$$P_{p,6}(z) = z^3 - (p-2)z^2 - (p-2)z + 1 = (z+1)(z^2 - (p-1)z + 1) :$$

which gives the second statement. □

Theorem 32. *Let $p, q \geq 3$ and q even and $G_{p,q}$ a p, q regular tessellation of negative curvature, i.e., $\kappa(x) = p\big(\frac{1}{p} - \frac{1}{2} + \frac{1}{q}\big) < 0$, $x \in X$. Denote by $\lambda_{\max}(p, q)$ the largest real root of $P_{p,q}$. Then,*

$$\mu = \log(\lambda_{\max}(p, q)).$$

The proof consists of several lemmas. For the rest of the section we consider a negatively curved p, q-regular tessellation with even q.

Let $x_0 \in X$ be fixed and let B_n be the balls about x_0.

For $l = 1, \ldots, q-1$ define

$$c_n^l = \#\{f \in \partial_F B_n \mid \deg_n(f) := q - \deg_{\mathrm{B_n}}(f) = \#\{x \in X \setminus B_n \cap f\} = l\},$$

Lemma 34. *For all $n \geq 0$*

(i) $c_n^l = c_{n-1}^{l+2}$ *for* $1 \leq l \leq q-3$

(ii) $c_n^{q-2} = c_{n-1}^2 = 0$

(iii) $c_n^{q-1} = c_n^1 + s_{n+1} - s_n$, *where* $s_k = |S_k|$.

(iv) $c_n^l = 0$ *for even* $1 \leq l \leq q-1$.

Proof. (i) If f is such that $\deg_n(f) = l$, $1 \leq l \leq q-3$, then $\deg_{n-1}(f) = l+2$.
(ii) $\deg_n(f) = q-2$, then $f \cap B_{n-1} = \emptyset$. However, f induces a 'horizontal' edge which gives rise to a unique $g \in F$ with $g \cap B_{n-1} \neq \emptyset$ and $\deg_n(g) = 2$.
(iii) A moments thought gives that s_{k+1} coincides with the number of faces in the boundary of B_k that have more than one vertex outside of B_k. Hence, by (i) and (ii)

$$\begin{aligned} s_{n+1} &= \sum_{l=2}^{q-1} c_{n-1}^l = c_{n-1}^2 + \sum_{l=3}^{q-1} c_{n-1}^l = c_n^{q-2} + \sum_{l=1}^{q-3} c_n^l = c_n^1 + \sum_{l=2}^{q-1} c_n^l - c_n^{q-1} \\ &= c_n^1 + s_{n+2} - c_n^{q-1} = c_{n-1}^3 + s_{n+2} - c_n^{q-1}. \end{aligned}$$

The last statement follows by induction over n. □

Lemma 35. *For all $n \geq 0$ and $\beta = \frac{q-2}{2q}$*

$$\kappa(B_n) = 1 - \beta(s_{n+1} - s_n) + \sum_{l=1}^{N-1} ((2l+1)\beta - l) c_n^{2l+1}.$$

Proof. We have $|\partial B_n| = \sum_{l=1}^{q-1} c_n^l$. Thus by Proposition 6 and (iii) from the lemma above.

$$\begin{aligned}\kappa(B_n) &= 1 - \frac{|\partial B_n|}{2} + \sum_{f \in \partial_F W} \frac{\deg_{B_n}(f)}{\deg(f)} \\ &= 1 - \sum_{l=1}^{q-1} \frac{1}{2} c_n^l + \frac{1}{q} \sum_{l=1}^{q-1} l c_n^l \\ &= 1 + \sum_{l=1}^{q-1} \frac{q-2l}{2q} c_n^l \\ &= 1 - \frac{q-2}{2q}(s_{n+1} - s_n) + \sum_{l=2}^{q-2} \frac{q-2l}{2q} c_n^l.\end{aligned}$$

By (iv) from the lemma above we have $c_n^l = 0$ for even l. Thus, since $N = \frac{q-2}{2}$

$$\sum_{l=2}^{q-2} \frac{q-2l}{2q} c_n^l = \sum_{l=3}^{q-3} \frac{q-2l}{2q} c_n^l = \sum_{l=1}^{N-1} \frac{q-2(2l+1)}{2q} c_n^{2l+1} = \sum_{l=1}^{N-1} ((2l+1)\beta - l) c_n^{2l+1}$$

□

Lemma 36. *We have*

$$s_{n+N+1} = (p-2) \sum_{k=0}^{N-1} s_{n+N-k} - s_n$$

Proof. By (iv) for even l we have $c_n^l = 0$. Applying (i) repeatedly we obtain with for l even

$$c_n^{2l+1} = c_{n-l}^1 = c_{n-l}^{q-1} - s_{n-l+1} + s_{n-l} = c_{n-N}^l - s_{n-l+1} + s_{n-l}$$

since $N = \frac{q-2}{2}$. Thus by the lemma above

$$\begin{aligned}\kappa(B_n) - 1 + \beta(s_{n+1} - s_n) &= \sum_{l=1}^{N-1} ((2l+1)\beta - l) c_n^{2l+1} \\ =& \sum_{l=1}^{N-1} ((2l+1)\beta - l)(c_{n-N}^{2l+1} - s_{n-l+1} + s_{n-l}) \\ =& \kappa(B_{n-N}) - 1 + \beta(s_{n-N+1} - s_{n-N}) - \sum_{l=1}^{N-1} ((2l+1)\beta - l)(s_{n-l+1} - s_{n-l})\end{aligned}$$

$$\begin{aligned}\sum_{l=0}^{N-1}\kappa(S_{n-l}) =& -\beta(s_{n+1}+s_{n-N})+(\beta-(3\beta-1))s_n \\ &+(\beta+(2(N-1)+1)\beta-(N-1))s_{n-N+1}-\sum_{l=1}^{N-2}(2\beta-1)s_{n-l} \\ =& -\beta(s_{n+1}+s_{n-N})+(1-2\beta)\sum_{l=0}^{N-1}s_{n-l}\end{aligned}$$

Since the graph is p,q regular we have $\kappa(x)=p(\frac{1}{p}-\frac{1}{2}+\frac{1}{q})=1-p\beta$ and the statement follows. □

Proof of Theorem 32. Let M the $(N+1)\times(N+1)$ matrix be given as

$$M=\begin{pmatrix}(p-2) & \dots & (p-2) & -1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0\end{pmatrix}$$

and $r_n(s_n,\dots,s_{n-N})$, $n\geq 1$. Then,

$$Mr_n=r_{n+1}.$$

The eigenvalues of M are given by the roots of the characteristic polynomial, which we calculate by expanding with respect to the first column, i.e., let $a=p-2$

$$\begin{aligned}&\det(M-z) \\ &=(a-z)\det\begin{pmatrix}-z & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & -z\end{pmatrix}-\det\begin{pmatrix}a & \cdots & a & -1 \\ 1 & -z & & \\ & \ddots & \ddots & \\ & & 1 & -z\end{pmatrix} \\ &=P_{p,q}(z).\end{aligned}$$

Thus the eigenvalues $\lambda_0,\dots,\lambda_{N+1}$ are given by the roots of $P_{p,q}$. By Proposition 8 there are $N-1$ roots $\lambda_1,\dots,\lambda_{N-1}$ which lie on the complex unit circle except for $\lambda_{N+1}=\lambda_{p,q}^{\max}>1$ and $\lambda_N=1/\lambda_{p,q}^{\max}$. Let $v_1=\overline{v_2},\dots,v_{N-2}=\overline{v_{N-1}},v_N,v_{N+1}$ be the corresponding eigenvectors. Thus, for all $n\geq 0$

$$r_{n+N}=M^n r_N=\sum_{j=1}^{N+1}\langle v_j,r_N\rangle\lambda_j^n v_j$$

Since by Lemma 36 the sequence (s_n) is strictly increasing for large n, we have that $r_{N+1}=(s_N,\dots,s_0)\not\perp v_N$ and, therefore,

$$s_{n+1}\sim(\lambda_{p,q}^{\max})^n.$$

It remains to check that $\limsup\frac{1}{n}\log s_n=\limsup\frac{1}{n}\log\#B_n$ which is left as an exercise. □

10.3 Cheeger constants of regular tessellations

Literature: Häggström, Jonasson and Lyons

Theorem 33. *Let $p, q \geq 3$ and $G_{p,q}$ be a p, q-regular tessellation of non-positive curvature, i.e., $\frac{1}{p} - \frac{1}{2} + \frac{1}{q} < 0$. Then,*

$$\alpha = \frac{p-2}{p}\sqrt{1 - \frac{4}{(p-2)(q-2)}}.$$

For the proof define two auxiliary constants for a tessellation G

$$\beta := \beta(G) := \lim_{N\to\infty} \inf\{\frac{|K|}{|E_K|} \mid K \subseteq X \text{ connected } N \leq |K| < \infty\},$$

$$\delta := \delta(G) := \lim_{N\to\infty} \sup\{\frac{|K|}{|E_K| + |\partial K|} \mid K \subseteq X \text{ connected } N \leq |K| < \infty\}$$

Lemma 37.

$$\alpha = \lim_{N\to\infty} \inf\{\frac{|\partial K|}{2|E_K| + |\partial K|} \mid K \subseteq X \text{ connected } N \leq |K| < \infty\}.$$

Proof. Benjamini,Lyons,Peres,Schramm '99 □

Lemma 38. *For all graphs with $\deg(x) = p$ with $p \in X$ we have*

$$p\beta = \frac{2}{1-\alpha} \quad \text{and} \quad p\delta = \frac{2}{1+\alpha}$$

In particular,

$$\beta = \inf\{\frac{|K|}{|E_K|} \mid K \subseteq X \text{ finite}\}, \quad \delta = \sup\{\frac{|K|}{|E_K| + |\partial K|} \mid K \subseteq X \text{ finite}\}.$$

Proof. For K finite we have $p|K| = 2|E_K| + |\partial K|$. Therefore,

$$\frac{2}{1 - \frac{|\partial K|}{2|E_K| + |\partial K|}} = \frac{2|E_K| + |\partial K|}{|E_K|} = p\frac{|K|}{|E_K|}$$

and

$$\frac{2}{1 + \frac{|\partial K|}{2|E_K| + |\partial K|}} = \frac{2|E_K| + |\partial K|}{|E_K| + |\partial K|} = p\frac{|K|}{|E_K| + |\partial K|}$$

Thus, the statements follow from Lemma 37. □

For a tessellation $G = (X, E, F)$ we define the dual tessellation $G^* = (X^*, E^*, F^*)$ by letting $X^* = F$, $F^* = X$ and $x^*, y^* \in X$ are joined by an edge in E^* if they share an edge as faces. We have $(G^*)^* = G$ and if $G_{p,q}$ is a p, q-regular tessellation then $G^*_{p,q}$ is a q, p-regular tessellation.

Theorem 33 follows from the following proposition.

Proposition 10. *Let G be a regular tessellation. Then, $\beta(G) + \delta(G^*) = 1$*

Proof. In the definition of α it suffices to consider simply connected sets since by filling the 'holes' the boundary becomes smaller and the volume becomes larger. By Lemma 38 this also holds for β. Next we pursue a similar strategy for $\delta(G^*)$. Let $K^* \in X^*$ which corresponds to a set of faces in G. Let $\overline{K}$ be the set of vertices contained in these faces. Let $\widehat{K}^*$ be the set of faces in enclosed by $E_{\overline{K}}$ in G which corresponds to a set of vertices in G^* which we also denote by $\widehat{K}^*$. Note that $|\widehat{K}^*| \geq |K^*|$ and $|\partial \widehat{K}^*| \leq |\partial K^*|$. Moreover, $|\partial \widehat{K}^*| + |E_{\overline{K}^*}| = |E_{\overline{K}}|$.

We show $\beta(G) + \delta(G^*) \leq 1$: Let $\varepsilon > 0$ and let $K^* \in V^*$

$$|\partial K^*| + |E_{K^*}| > \frac{1}{\varepsilon}$$
$$\frac{|K^*|}{|\partial K^*| + |E_{K^*}|} \geq \delta(G^*) - \varepsilon$$
$$|E_{K^*}| + |\partial K^*| = |E_{\overline{K}}|$$

which is possible by what we discussed above. For the number of faces in $G_{\overline{K}} = (\overline{K}, E_{\overline{K}}, F_{\overline{K}})$ we have $|F_{\overline{K}}| \geq |K^*| + 1$ (inequality as not necessarily simply connected and 1 for the unbounded face outside). Thus, Euler's formula with respect to $G_{\overline{K}}$ gives

$$\frac{|\overline{K}|}{|E_{\overline{K}}|} + \frac{|K^*|}{|\partial K^*| + |E_{K^*}|} \leq \frac{|\overline{K}| + |F_{\overline{K}}| - 1}{|\partial K^*| + |E_{K^*}|} \leq 1 + \frac{1}{|\partial K^*| + |E_{K^*}|} < 1 + \varepsilon$$

Since we chose K^* such that $\frac{|K^*|}{|\partial K^*| + |E_{K^*}|} \geq \delta(G^*) - \varepsilon$,

$$\frac{|\overline{K}|}{|E_{\overline{K}}|} + \delta(G^*) \leq 1 + 2\varepsilon.$$

As $G_{\overline{K}}$ is connected and $|\overline{K}| \to \infty$ when $\varepsilon \to 0$, it follows $\beta(G) + \delta(G^*) \leq 1$.
We next show $\beta(G) + \delta(G^*) \geq 1$: Let $\varepsilon > 0$ and $K \subseteq X$ simply connected such that

$$\frac{|K|}{|E_K|} \leq \beta(G) + \varepsilon.$$

Let K^* be the vertices in G^* corresponding to the faces in G_K. Since $|\partial K^*| + |E_{K^*}| \leq |E_K|$ (why inequality?) and the number of faces $|F_K|$ of G_K is equal to $|K^*| + 1$ (the one is for the unbounded face outside)

$$\frac{|K|}{|E_K|} + \frac{|K^*|}{|\partial K^*| + |E_{K^*}|} \geq \frac{|K| + |F_K| - 1}{|E_K|} = 1 + \frac{1}{|E_K|} \geq 1$$

by Euler's formula applied to G_K. By Lemma 38 the supremum in δ can also be taken over all finite sets and by our choice of K, i.e., $\frac{|K|}{|E_K|} \leq \beta(G) + \varepsilon$,

$$\beta(G) + \delta(G^*) + \varepsilon \geq \frac{|K|}{|E_K|} + \frac{|K^*|}{|\partial K^*| + |E_{K^*}|} \geq 1.$$

Since $\varepsilon > 0$ is arbitrary the statement follows. □

Proof of Theorem 33. By the proposition above we have $\beta(G) + \delta(G^*) = 1$ and $\beta(G^*) + \delta(G) = 1$ and by Lemma 38

$$1 = \beta(G) + \delta(G^*) = \frac{2}{p(1-\alpha(G))} + \frac{2}{q(1+\alpha(G^*))} = 2\frac{q + q\alpha(G^*) + p - p\alpha(G)}{pq(1-\alpha(G))(1+\alpha(G^*))}$$
$$1 = \beta(G^*) + \delta(G) = \frac{2}{q(1-\alpha(G^*))} + \frac{2}{p(1+\alpha(G))} = 2\frac{p + p\alpha(G) + q - q\alpha(G^*)}{pq(1-\alpha(G^*))(1+\alpha(G))}$$

and

$$pq(1 + \alpha(G^*) - \alpha(G) - \alpha(G)\alpha(G^*)) = 2(p - p\alpha(G) + q + q\alpha(G^*))$$
$$pq(1 - \alpha(G^*) + \alpha(G) - \alpha(G)\alpha(G^*)) = 2(p + p\alpha(G) + q - q\alpha(G^*))$$

and

$$1 - \frac{2(p+q)}{pq} = \alpha(G)\alpha(G^*)$$
$$q(p-2)\alpha(G^*) = p(q-2)\alpha(G)$$

Thus,

$$\alpha(G) = \sqrt{\frac{pq - 2p - 2q}{p^2}\frac{(p-2)}{(q-2)}} = \frac{(p-2)}{p}\sqrt{1 - \frac{4}{(p-2)(q-2)}}$$

□

10.4 Absence of essential spectrum of rapidly growing tessellations

In this section we show that if a tessellation has uniformly unbounded curvature then the corresponding Laplacian has purely discrete spectrum.

Let

$$\kappa_\infty := \lim_{n\to\infty} \sup_{x \in X \setminus B_n} \kappa(x) = -\infty.$$

Theorem 34. *$\sigma_{\mathrm{ess}}(\Delta) = \emptyset$ if and only if κ_∞.*

Recall that a set $W \subseteq X$ is called *simply connected* if W and $X \setminus W$ are connected.

Lemma 39. *Let $a_1, \ldots, a_m > 0$ and $b_1, \ldots, b_m > 0$. Then,*

$$\min_{i=1,\ldots,m} \frac{a_i}{b_i} \leq \frac{\sum_{i=1}^m a_i}{\sum_{i=1}^m b_i} \leq \max_{i=1,\ldots,m} \frac{a_i}{b_i}$$

Proof. Assume $a_1/b_1 \geq \ldots \geq a_m/b_m$

$$\frac{\sum_{i=1}^m a_i}{\sum_{i=1}^m b_i} - \frac{a_m}{b_m} = \frac{\sum_{i=1}^m (a_i b_m - a_m b_i)}{\sum_{i=1}^m b_i b_m} \geq 0$$

The other inequality is proven analogously. □

Lemma 40. *For any finite set $W \subseteq X$ there is a finite simply connected set $U \subseteq X$ such that*

$$\frac{|\partial W|}{n(W)} \geq \frac{|\partial U|}{n(U)}.$$

In particular, it suffices to consider simply connected sets in the infimum of the Cheeger constant.

Proof. If W is not connected we choose the component U with the smallest ratio $|\partial U|/n(U)$ and the statement follows from the lemma above. Suppose $X \setminus U$ is not connected. Since the graph is locally finite there are at most finitely many components and by Jordan's curve theorem there is at most one infinite component. Let $W_1, \ldots, W_n$ be the finite components of $X \setminus U$. Now $X \setminus U$ with $V = U \cup W_1 \cup \ldots \cup W_n$ is connected and since $\partial V \subseteq \partial U$ and $n(U) \leq n(V)$ (as $U \subseteq V$) we have $|\partial U|/n(U) \geq |\partial V|/n(V)$. □

Theorem 35. *Let $U \subseteq X$. Then,*

$$\alpha_U \geq -2 \sup_{x \in U} \frac{1}{\deg(x)} \kappa(x)$$

In particular, if $\deg > 6$ *then* $\alpha_U > 0$.

Proof. If $\kappa(x) = 0$ for some $x \in U$, there is nothing to prove. So, assume $\kappa(x) > 0$ for all $x \in U$. By Proposition 6 we have for all finite and simply connected $W \subseteq U \subseteq X$

$$\kappa(W) = 1 - \frac{|\partial W|}{2} + \sum_{f \in \partial_F W} \frac{\deg_W(f)}{\deg(f)} \geq -\frac{|\partial W|}{2}$$

Since $\kappa(x) < 0$, we get by the inequality above and Lemma 39,

$$\frac{|\partial W|}{n(W)} \geq -\frac{2\kappa(W)}{n(W)} \geq -\frac{\sum_{x \in W} 2\kappa(x)}{\sum_{x \in W} \deg(x)} \geq -2 \sup_{x \in U} \frac{1}{\deg(x)} \kappa(x).$$

□

Proof of Theorem . Note that

$$1 - \frac{\deg(x)}{2} \leq \kappa(x) \leq 1 - \frac{\deg(x)}{6}.$$

Thus, for (x_n) in X, $\kappa(x_n) \to -\infty$ if and only if $\deg(x_n) \to \infty$. By Lemma 35 the assumption $\kappa_\infty = -\infty$ implies $\alpha_\infty > 0$. Now, the statement follows from Corollary 12. $\square$