Absolutely continuous spectrum on trees: random potentials, random hopping and Galton-Watson trees

Matthias Keller
(joint work with Daniel Lenz and Simone Warzel)

We study operators on rooted trees with an underlying substitution structure. These trees are often called trees of finite cone type and their graph Laplacians exhibit finitely many bands of purely absolutely continuous spectrum. This absolutely continuous spectrum is shown to be stable under various random perturbations - small but extensive as well as large but rare. These include small random potentials and hopping terms and on the other hand multi-type Galton-Watson trees with a distribution close to a deterministic one.

Trees of finite cone type are defined by a finite set $\mathcal{A}$ and a matrix $M : \mathcal{A}\times\mathcal{A} \to \mathbb{N}$. To each $j \in \mathcal{A}$ we associate a rooted tree $\mathbb{T} = \mathbb{T}(M, j)$ with root $o = o(j)$ and labeling in $\mathcal{A}$ by the following rules: The root carries label j and each vertex of label k has $M_{k,l}$ forward neighbors of label l for $k, l \in \mathcal{A}$.

We consider the Laplacian $\Delta : \ell^2(\mathbb{T}) \to \ell^2(\mathbb{T})$ with a boundary condition at the root

$$\Delta\varphi(x) = \sum_{y\sim x}(\varphi(x) - \varphi(y)) + 1_{\{x=o\}}\varphi(o).$$

It is well known that the Green functions $z \mapsto G_x(z, \Delta) = \langle\delta_x, (\Delta - z)^{-1}\delta_x\rangle$, $x \in \mathbb{T}$, are analytic functions from the upper half plane $\mathbb{H}$ into itself. In order to study the spectrum $\sigma(\Delta)$ of Δ, we investigate the Green functions in the limits $\Im z \downarrow 0$. For the Laplacian we get the following:

Theorem 1 ([5, 7]). *For all $x \in \mathbb{T}$, the functions $E \mapsto G_x(E + i\varepsilon, \Delta)$ stay uniformly bounded as $\varepsilon \downarrow 0$ and there is a finite set $\Sigma_0 \subset \mathbb{R}$ such that $G_x(E + i0, \Delta) = \lim_{\varepsilon\downarrow 0} G_x(E + i\varepsilon, \Delta)$ exists all for $E \in \mathbb{R} \setminus \Sigma_0$. Moreover, the function*

$$\mathbb{R} \setminus \Sigma_0 \to \mathbb{R} \cup \mathbb{H}, \quad E \mapsto G_x(E + i0, \Delta)$$

is continuous and takes values in $\mathbb{H}$ on finitely many intervals. In particular, $\sigma(\Delta)$ consists of finitely many bands of purely absolutely continuous spectrum.

The absolutely continuous spectrum on non-regular trees turns out to be very stable under radially symmetric perturbations.

Theorem 2 ([5, 7]). *Let $\mathbb{T}$ be non-regular and $I \subset \sigma(\Delta) \setminus \Sigma_0$ be compact. Then there is $\lambda > 0$ such that for all radially symmetric $v : \mathbb{T} \to [-\lambda, \lambda]$ the map*

$$I \to \mathbb{H}, \quad E \mapsto G_x(E + i0, \Delta + v)$$

is continuous. In particular, the spectrum of $\Delta + v$ is purely absolutely continuous on I. If additionally $v(x) \to 0$ as $|x| \to \infty$, then $\sigma_{ac}(\Delta) = \sigma_{ac}(\Delta + v)$.

We now turn to the stability of the absolutely continuous spectrum under random perturbations. In particular, we look at three models: random potentials, random hopping and multi-type Galton-Watson trees.

(a) **Random potentials.** Let $(v_x)_{x\in\mathbb{T}}$ be independent identically distributed random variables with support in $[-1,1]$. For $\lambda \geq 0$, let

$$H^\omega = \Delta + \lambda v^\omega, \qquad \omega \in \Omega.$$

Stability of absolutely continuous spectrum for small λ on regular trees was first proven in [9] which was followed by [1, 2, 3].

(b) **Random hopping.** Let $(t_{xy})_{x\sim y}$ be independent identically distributed random variables on the edges with support in $(-1,1)$. Then, for $\lambda \in [0,1]$ the operators $T^\omega : \ell^2(\mathbb{T}) \to \ell^2(\mathbb{T})$ given by

$$T^\omega \varphi(x) = \sum_{y\sim x}(1+\lambda t^\omega_{xy})(\varphi(x)-\varphi(y))$$

define the random hopping model. It sometimes goes under the name of first passage percolation, where transit times of random walks are studied.

(c) **Multi-type Galton-Watson trees.** Finally, we consider a randomization of the geometry. Let b be a multi-type Galton-Watson branching process with types in $\mathcal{A}$. Denote the set of realizations by Θ. We are interested in the absolutely continuous spectrum of the operators $\Delta^\theta : \ell^2(\theta) \to \ell^2(\theta)$ given by

$$\Delta^\theta \varphi(x) = \sum_{y\sim x}(\varphi(x)-\varphi(y)), \qquad \theta \in \Theta.$$

For $s \in \mathbb{N}_0^{\mathcal{A}}$, $j \in \mathcal{A}$, denote by $\mathbb{P}_j^{(b)}(s)$ the probability that a vertex of label j has s_k, $k \in \mathcal{A}$, forward neighbors. We impose two assumptions on b:

(i) Every vertex has at least one forward neighbor: $\mathbb{P}_j^{(b)}(s \equiv 0) = 0$, $j \in \mathcal{A}$.

(ii) $\sum_{s\in\mathbb{N}_0^{\mathcal{A}}} \mathbb{P}_j^{(b)}(s)\|s\|^2 < \infty$, $j \in \mathcal{A}$, where $\|s\| = \sum_{k\in\mathcal{A}} s_k$.

Note that in order to expect purely absolutely continuous spectrum, assumption (i) is vital. Dropping (i) immediately yields eigenvalues with compactly supported eigenfunctions spread all over the spectrum. Furthermore, for b_1, b_2 satisfying (ii) we can define the metric

$$d(b_1,b_2) = \max_{j\in\mathcal{A}} \sum_{s\in\mathbb{N}_0^{\mathcal{A}}} \left|\mathbb{P}_j^{(b_1)}(s) - \mathbb{P}_j^{(b_2)}(s)\right| \|s\|^2.$$

If a process b satisfies $\mathbb{P}_j^{(b)}(s = M_{j,\cdot}) = 1$ for all $j \in \mathcal{A}$, then the set of realizations consists exactly of the elements $\mathbb{T}$ which are given by substitution matrix M. In this case, we denote $b = b_M$. Apart from the deterministic case, the simplest example is the one of a binary tree, where one of the forward edges of each vertex is deleted with probability $1-p$, $p \in (0,1)$. For small p this model is discussed in [4].

For the models (a), (b) and (c) we have the following theorem:

Theorem 3. *Let $I \subset \sigma(\Delta) \setminus \Sigma_0$ be compact. There is $\lambda > 0$ such that*

(a) ([5, 8]) *H^ω has purely absolutely continuous spectrum in I a.s.,*

(b) ([5, 8]) *T^ω has purely absolutely continuous spectrum in I a.s.,*

(c) ([6]) Δ^θ *has purely absolutely continuous spectrum in* I *for a.e.* $\theta \in \Theta^b$ *whenever* b *is such that* $d(b, b_M) < \lambda$ *for some* M*.*

References

[1] M. Aizenman, R. Sims, and S. Warzel, *Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs.* Probability Theory and Related Fields **136** (2006), 363–394.

[2] M. Aizenman, and S. Warzel, *Resonant delocalization for random Schrödinger operators on tree graphs.* Preprint (2011), arXiv:1104.0969.

[3] R. Froese, D. Hasler, and W. Spitzer, *Absolutely continuous spectrum for the Anderson model on a tree: a geometric proof of Klein's theorem.* Comm. Math. Phys. **269** (2007), 239–257.

[4] R. Froese, D. Hasler, and W. Spitzer, *A geometric approach to absolutely continuous spectrum for discrete Schrödinger operators.* Boundaries and Spectra of Random Walks, Progress in Probability, Birkhaeuser, 2011.

[5] M. Keller, *On the spectral theory of operators on trees.* Ph.D. Thesis 2010.

[6] M. Keller, *Absolutely continuous spectrum for multi-type Galton Watson trees.* Preprint (2011), arXiv:1109.1959.

[7] M. Keller, D. Lenz, and S. Warzel, *On the spectral theory of trees with finite cone type.* To appear in Israel Journal of Mathematics, arXiv:1001.3600.

[8] M. Keller, D. Lenz, and S. Warzel, *Absolutely continuous spectrum for random operators on trees of finite cone type.* To appear in Journal d'Analyse Mathematique, arXiv:1108.0057

[9] A. Klein, *Absolutely continuous spectrum in the Anderson model on the Bethe lattice.* Math. Res. Lett. **1** (1994), 399–407.